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Type-II matrices in weighted Bose-Mesner algebras of ranks 2 and 3

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Abstract Type-II matrices are nonzero complex matrices that were introduced in connection with spin models for link invariants. Type-II matrices have been found in connection with symmetric designs, sets of equiangular lines, strongly regular graphs, and some distance regular graphs. We investigate weighted complete and strongly regular graphs, and show that type-II matrices arise in this setting as well.

Keywords Type-II Matrix, Association Scheme, Spin Model

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1 Introduction

Spin models were introduced by V. Jones in [11] to construct link invariants. Nomura ([13]) found that matrices satisfying the type-II condition of Jones had nice properties, in particular that a Bose-Mesner algebra (now known as the Nomura algebra) could be constructed from any such type-II matrix. Because a spin model is contained in its Nomura algebra, it is natural to consider type-II matrices within the Bose-Mesner algebras of known association schemes. Chan and Godsil in [5] investigated the strongly regular graphs and found that up to 6 type-II matrices are found in their Bose-Mesner algebras. Further, they showed that type-II matrices arise in connection with other combinatorial structures, such as symmetric designs, sets of equiangular lines, and antipodal distance regular graphs with diameter 3.

The goal of this paper is to demonstrate that type-II matrices are also found in conjunction with certain weights that are regular (in the sense of Higman in [8]) on association schemes of rank 2 or 3. These are schemes in which the base graphs have edges weighted by ±1, and satisfy suitable conditions to make the linear span of the weighted adjacency matrices a semi-simple algebra.

In Section 2, we define the terms necessary to make the previous paragraph intelligible. We have benefited from accessible treatments of this introductory material in [10], [14], and [5]. First, we look at spin models and the closely related type-II matrices. Then, we define association schemes, which are essentially synonymous with Bose-Mesner algebras. We look next

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at the connection between type-II matrices and association schemes, via the Nomura algebra. In the last part of this section, we define regular weights on association schemes. Section 3 is a discussion of the type-II matrices found in the rank 2 case, where a regular weight with values ± 1 is equivalent to a regular 2-graph. In Section 4, the rank 3 case is treated. Here the association scheme is a complementary pair of strongly regular graphs. The general state of affairs is given and the remainder of the section is devoted to more explicit results pertaining to the lattice graph family of strongly regular graphs. Section 6 contains additional examples of ranks 2 and 3.

2 Preliminaries

To define spin models we require the notion of a Schur inverse. We will use the term 'Schur product' to denote entry-wise multiplication of matrices, also called the Hadamard product. A matrix W with nonzero entries is *Schur invertible*, meaning it has an inverse $W^{(-)}$ with respect to the Schur product. That is, $W \circ W^{(-)} = J$ where J is the all ones matrix.

Definition. An n by n complex matrix W is called a *type-II matrix* if it is Schur invertible and

$$WW^{(-)T} = nI.$$

Examples. The following are well-known type-II matrices:

1. The character table of an abelian group;
2. Hadamard matrices: entries are ± 1 and $HH^T = nI$;
3. tensors of type-II matrices;
4. the Potts model: set $\alpha = -\beta^{-3}$ where β satisfies $\beta^2 + \beta^{-2} + \sqrt{n} = 0$. $W := \alpha I + \beta(J - I)$ is a type-II matrix.

A type-II matrix may be obtained from another by *scaling* and/or by *permutation*. That is, if W is type II, Δ and Δ' are invertible diagonal matrices, and P and P' are permutation matrices, then $\Delta W \Delta'$ and PWP' are type-II matrices.

A *spin model* is a type-II matrix that satisfies an additional ("type III") condition which presents itself in a natural way via the Nomura algebra we define in the next section.

2.1 The Nomura algebra

Let W be a type-II matrix. Define column vectors ([13]):

$$\begin{aligned}(W_{i/j})_x &= W_{x,i} \cdot W^{(-)}_{x,j} \\ &= \frac{W_{x,i}}{W_{x,j}}.\end{aligned}$$

Observe that this is just the Schur ratio of columns i and j which will of course be the all ones vector when $i = j$.

Definition. The *Nomura algebra* of a Schur-invertible square matrix W is

$$\mathcal{N}_W := \{M \in \mathbb{C}_{n \times n} \mid W_{i/j} \text{ is an eigenvector of } M, \text{for all } i, j\}$$

The subscript will be suppressed when clarity allows. We have the following properties of $\mathcal{N}$ ([13]):

1. $\mathcal{N}$ is a matrix algebra.
2. $I \in \mathcal{N}$.
3. W is type II $\iff J \in \mathcal{N}$.
4. W is type II $\implies \mathcal{N}$ is commutative.
5. W is a spin model $\iff cW \in \mathcal{N}$ for some nonzero scalar c.

Example. For the Potts model, $W_{i/j}$ $(i \neq j)$ is a vector with entries α/β, β/α, and $n-2$ 1's. Since $W_{i/j}$ is an eigenvector of W, $W \in \mathcal{N}$ thus W is a spin model.

2.2 Association schemes

Let X be a finite set. Define a set of relations on X as matrices $\{A_i\}$ with rows and columns indexed by X, where i ranges over the indexing set $\mathbb{I} := \{0, 1, \ldots, d\}$.

Definition. A *d-class association scheme* $\mathcal{X} = (X, \{A_i\}, \mathbb{I})$ is a finite set X together with a set of $(0,1)$ matrices $\{A_i\}$ indexed by the set $\mathbb{I}$, satisfying:

1. $\sum_{i \in \mathbb{I}} A_i = J$;
2. $A_0 = I$;
3. $A_i^T = A_{i^*}$ for some $i^* \in \mathbb{I}$;
4. $A_i A_j = A_j A_i$;
5. $A_i A_j = \sum_k p_{ij}^k A_k$.

This last condition says that the linear span of the A_i over $\mathbb{C}$ is closed under multiplication. Thus $\mathcal{A} := \langle A_i \rangle$ forms a commutative, associative matrix algebra called the Bose-Mesner algebra of the scheme. The *intersection numbers* are the constants p_{ij}^k defined by (5). There is combinatorial significance to these which we do not make use of here. The interested reader should see [2], [4], [6] for more. The *intersection matrices*,

$$M_j := \left(p_{ij}^k\right)_{i,k \in \mathbb{I}} \qquad (j \in \mathbb{I})$$

store these numbers. More importantly, the regular representation $A_j \mapsto M_j$ is an isomorphism of (commutative) associative algebras; we take advantage of this fact to calculate the character-multiplicity table of $\mathcal{A}$. That is, $\mathcal{A}$ is semi-simple, thus decomposes into a direct sum of simple ideals. Commutativity ensures that the irreducible constituents of the standard character are linear, the end result being that eigenvalues and their multiplicities for $\{A_j\}$ may be computed from $\{M_j\}$.

The simplest association scheme is the 1-class scheme, consisting of I and $J-I$. As the latter is the adjacency matrix of the complete graph on X, we see that a 1-class scheme is equivalent to K_n. The *rank* of a scheme is $d+1$, the number of classes including the trivial one.

An association scheme with 2 classes consists of a strongly regular graph and its complement, along with the identity. This may in fact be taken as the definition of a strongly regular graph, but the usual definition follows.

Definition. A *strongly regular graph* (SRG) is a regular graph, neither complete nor null, with the number of vertices adjacent to two given vertices, x and y, depending only on whether or not x and y are adjacent.

In the context of an SRG, $n = |X|$ will denote the number of vertices, k the valency, and $l = n - k - 1$ the valency of the complement. The parameters of an SRG are usually given as $(n, k = p_{11}^0, p_{11}^1, p_{11}^2)$, all others being determined by these. Below we shall consider the pentagon, the triangular graphs, and the lattice graphs, all of which are strongly regular.

2.3 Theorem of Nomura

The following theorem of Nomura and Jaeger et. al. ([13], [10]) establishes a connection between type-II matrices and association schemes.

Theorem 2.1. *If W is a type-II matrix, $\mathcal{N}_W$ is the Bose-Mesner algebra of an association scheme.*
This means that $\mathcal{N}_W$ is a commutative matrix algebra containing I and J, closed under the Schur product, with a basis $\{A_i\}$ of $(0,1)$ matrices as in the previous section. This result is important as it tells us that every spin model can be found inside a Bose-Mesner algebra. As pointed out in [5], there are finitely many association schemes for a given n.

2.4 Weights

In this section we define regular weights on association schemes, which were introduced by D.G. Higman ([8]) in the more general context of coherent configurations. The intent here is to show that these constitute a nesting ground for type-II matrices that are generally not spin models. The focus of this work is weights of full rank with values ± 1 on schemes of rank 2 and 3. Those of rank 2 are equivalent to *regular 2-graphs* which have been investigated by Taylor, Seidel, Bussemaker and others ([21], [17], [20], [19], [3]).

Definition. A *2-graph* (X, Δ) is a set X of vertices and a subset Δ of the triples from X, called odd triples, such that every 4-set contains an even number of odd triples.

A 2-graph is *regular* if each pair of vertices is in a fixed number of odd triples. We refer to a regular 2-graph by its parameters (n, a), where n is the number of vertices and a is the number of odd triples containing a given pair.

From any simple graph Γ we may construct a 2-graph by designating the triples with an odd number of edges from Γ to be the odd triples. As Δ is invariant under Seidel switching—interchanging adjacencies and non-adjacencies for any vertex—we see that a 2-graph may be viewed as a switching class of graphs. In fact, these two sets are in one-to-one correspondence.

We form the *Seidel matrix* of a graph Γ by setting the (x, y) entry to be 0 if $x = y$, -1 if x is adjacent to y, and 1 otherwise. Then switching Γ on a subset of the vertices is accomplished via a similarity transform by a diagonal matrix with ± 1 on the diagonal. Thus switching-equivalent graphs have the same spectrum, so the spectrum of a 2-graph is well-defined.

The Seidel matrix of a graph may also be interpreted as a weight (with values ± 1) on the edges of the complete graph. In this interpretation, Higman's notion of a regular weight generalizes the regular 2-graph.

Let U_t be the set of complex t^{th} roots of unity. A *weight with values in U_t* is a 2-cochain $\omega \in C^2(X, U_t)$, which we will view as a matrix with rows and columns indexed by X. In particular, this means that ω is Hermitian with unit diagonal, and that $\omega^{(-)T} = \omega$. For the present paper we work with $t = 2$, with one exception.

The *support* of a matrix M, $\operatorname{supp}(M)$, is the set of indices on which it is nonzero. Weights with values in U_t are thus considered to have *full support*.

The standard coboundary operator defines a 3-cochain $\delta\omega$, giving a weight to each triple of points, or triangle, by

$$\delta\omega(x, y, z) = \omega(x, y)\overline{\omega(x, z)}\omega(y, z).$$

We next define regularity of a weight ω on an association scheme $\mathcal{X}$. It will be convenient to refer to a triple of points $(x, y, z) \in X^3$ as a *triangle of type* ${}^{k}_{ij}$ if (x, y) belongs to class i

(equivalently, $A_i(x,y)=1$), (y,z) belongs to class j, and (x,z) to class k. The weight of triangle (x,y,x) is $\delta\omega(x,y,z)$. Next, for (x,z) in class k, we put

$$\beta_{ij}(x,z,\alpha) := \left|\left\{y \mid (x,y,z) \text{ has type } {}^{k}_{ij} \text{ and weight } \alpha\right\}\right|.$$

The weight ω is *regular on* $\mathcal{X}$ if, for fixed i, j, and α, $\beta_{ij}(x,z,\alpha)$ depends only on the class k, not the choice of (x,z) in that class. In this case we write $\beta^k_{ij}(\alpha)$.

The *switching class* of ω is the set of matrices obtained via similarity transform by a diagonal matrix with entries in U_t. The importance of this is that switching-equivalent weights have the same coboundary, hence a regular weight is unique up to switching.

Observe that a regular weight with values ± 1 on the 1-class scheme given by K_n represents a regular 2-graph, with $\beta^1_{11}(-1)$ the number of odd triples containing a given pair of vertices.

Define the weighted adjacency matrices

$$A_i^\omega := \omega \circ A_i,$$

noting that ω is just the sum of these. It follows from regularity that the A_i^ω span a matrix algebra with structure constants β^k_{ij}. That is,

$$A_i^\omega A_j^\omega = \sum_k \beta^k_{ij} A_k^\omega, \quad \text{where } \beta^k_{ij} = \sum_\alpha \alpha \beta^k_{ij}(\alpha).$$

Observe that β^k_{ij} is the sum of the weights of all triangles of type ${}^{k}_{ij}$ for a fixed (x,z) in class k. Note also that $\sum_\alpha \beta^k_{ij}(\alpha) = p^k_{ij}$. The weighted intersection numbers are therefore bounded in absolute value by the ordinary intersection numbers.

The weighted Bose-Mesner algebra $\mathcal{A}^\omega := \langle A_i^\omega \rangle$ is a semi-simple associative algebra, commutative if the scheme is symmetric and the weight real-valued. Again, the regular representation $A_j^\omega \mapsto M_j^\omega := \left(\beta^k_{ij}\right)_{i,j\in\mathbb{I}}$ is an isomorphism of associative algebras. The *rank* of a weight is the number of indices i for which A_i^ω is nonzero. Since our weights have full support, this coincides with the rank of the underlying association scheme. In the present paper, the schemes will be rank 2 and 3 and symmetric. Hence the A_i^ω are simultaneously diagonalizable with eigenvalues and multiplicities determined from M_i^ω.

2.5 Trivial weights

A regular weight ω on a scheme $\mathcal{X}$ will be called *trivial* if $A_i^\omega = c_i A_i$ with $c_i \in \mathbb{C}$, for all i, as this implies $\mathcal{A}^\omega = \mathcal{A}$. It is possible that a regular weight ω on $\mathcal{X}$ is trivial on some fission scheme of $\mathcal{X}$, or more generally (remove axiom 4 and replace 2 by: $\sum_{i\in\Omega} A_i = I, \Omega \subseteq \mathbb{I}$) a *coherent configuration* that is a fission of $\mathcal{X}$. A coherent configuration (CC) is *homogeneous* if $|\Omega| = 1$; an association scheme is therefore a homogeneous, commutative CC. The algebra $\langle A_i \rangle_{\mathbb{C}}$ is called the *coherent* (or *cellular*) *algebra* of the configuration ([7], [12]).

Definition. The *coherent closure* of a matrix X, $\mathrm{ccl}(X)$, is the intersection of all coherent algebras containing X.

This is well-defined because the intersection of two coherent algebras is coherent, and is nonempty because the full matrix algebra $M_n(\mathbb{C})$ is coherent and contains X.

Proposition 2.1. *Let ω be a regular weight on $\mathcal{X} = (X, \{A_i\})$. Setting $\mathcal{B}$:=ccl(ω) and letting $\mathcal{Y} := (X, \{B_i\})$ be the underlying CC, we have:*

1. *ω is trivial on $\mathcal{Y}$;*

2. ω is trivial on a CC $\mathcal{Z}$ if and only if $\mathcal{Z}$ is a fission of $\mathcal{Y}$.

Proof. 1. Define matrices $A_i^\omega(\alpha)$ by

$$(A_i^\omega(\alpha))_{x,y} := \begin{cases} 1 & (A_i^\omega)_{x,y} = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

so that $A_i^\omega = \sum \alpha A_i^\omega(\alpha)$. For each i and α, ccl(ω) contains $A_i^\omega(\alpha)$, by the Schur-Wielandt principle (6.5 in [12]). Write $A_i^\omega(\alpha) = \sum_j c_j B_j$. Because the B_j's have non-overlapping supports, and entries on both sides of the equation are 0 or 1, $c_j = 0$ or 1 for all j. Recalling that

$$\sum_{i,\alpha} A_i^\omega(\alpha) = J = \sum_j B_j,$$

we find that for each j, there exists i such that $A_i^\omega(\alpha) \circ B_j \neq 0$. This implies $c_j = 1$ in the expression for $A_i^\omega(\alpha)$, and $A_i^\omega(\alpha) \circ B_j = B_j$. Thus $\text{supp}(B_j) \subseteq \text{supp}(A_i^\omega(\alpha))$. We conclude that ω is constant on B_j, hence trivial on $\mathcal{Y}$.

2. Suppose ω is trivial on $\mathcal{Z} = (X, \{C_i\})$ with coherent algebra $\mathcal{C}$. Then $\omega \circ C_i = \alpha_i C_i$ for some $\alpha_i \in U_t$. We have

$$\omega = \omega \circ J = \sum_i \omega \circ C_i = \sum_i \alpha_i C_i \in \mathcal{C}.$$

Now $\mathcal{B} \subseteq \mathcal{C}$ by definition of coherent closure. It follows that $\mathcal{Z}$ is a fission of $\mathcal{Y}$.
The other direction is immediate.

□

Remark.
Switching does not fix ccl(ω). For example, five weights in the switching class of the Petersen graph can be found with coherent closures of dimensions 3, 6, 15, 18, and 22.

Theorem 2.2. *Let $W = \sum_i \alpha_i A_i^\omega$ be a type-II matrix in $\mathcal{A}^\omega$ where ω is regular with values in U_t, $\alpha_i \in \mathbb{C}$, and $\alpha_i^t \neq \alpha_j^t$ for $i \neq j$. Then W is a spin model if and only if $\mathcal{A}^\omega \subseteq \mathcal{N}_W$.*

Proof. Suppose $W \in \mathcal{N} = \mathcal{N}_W$. Since $\mathcal{N}$ is a coherent algebra, ccl$(W) \subseteq \mathcal{N}$. W has distinct entries on the supports of A_i^ω and A_j^ω for $i \neq j$ because for $u_i, u_j \in U_t$,

$$\alpha_i u_i = \alpha_j u_j \implies \frac{\alpha_i}{\alpha_j} \in U_t.$$

By the Schur-Wielandt principle, $A_i^\omega \in \text{ccl}(W) \subseteq \mathcal{N}$ for all i, hence $\mathcal{A}^\omega \subseteq \mathcal{N}$. □

2.6 Type-II matrices in weighted Bose-Mesner algebras

As mentioned, two regular weights ω_1 and ω_2 on the association scheme $\mathcal{A}$ may be equivalent under switching, meaning multiplying on the left and the right by a diagonal matrix. In the type-II matrix literature, this is generally referred to as *scaling* and is not restricted to ± 1's (nor to operating in the same way on rows and columns). However, for our purposes the scaling matrix will always have entries ± 1 on the diagonal.

Switching changes the edge weights, but has no effect on the basic graphs of the underlying association scheme. Let ω and ω' be switching-equivalent weights so that $\omega' = D\omega D$ for some diagonal matrix D. The algebra $\mathcal{A}^{\omega'}$ is isomorphic to $\mathcal{A}^\omega$ and both have $\mathcal{A}$ as the Bose-Mesner algebra of the underlying scheme. For any type-II matrix W in $\mathcal{A}^\omega$, the equivalent type-II matrix $W' := DWD$ is contained in $\mathcal{A}^{\omega'}$. All of this is completely straight-forward. Now, consider the Nomura algebras for W and W'. Scaling almost preserves the vectors $W_{i/j}$. Specifically, it preserves them up to multiplication by -1. Hence the Nomura algebras $\mathcal{N}_W$ and $\mathcal{N}_{W'}$ are *identical*. It can happen, and we give an explicit example in the next section, that W' is contained in $\mathcal{N}_{W'} = \mathcal{N}_W$ while W is not. A type-II matrix W is a spin model if and only if $cW \in \mathcal{N}_W$. Technically, then, W' is a spin model while W is not, even though they are equivalent.

The question of whether a given type-II matrix is equivalent to a spin model is therefore not a trivial one, even when the explicit vectors $W_{i/j}$ have been calculated. One method, for which we are grateful to a referee, is to use the modular invariance equation (see Prop. 12 in [10] and Section 3.2 in [1]) to test for spin models inside $\mathcal{N}_W$, and if one is found determine whether it is equivalent to W.

If a weight ω is regular on a scheme $\mathcal{X} = (X, \{A_i\})$ with values in U_t, then

$$(A_i^\omega)^{(-)T} = (\overline{\omega \circ A_i})^T = \overline{\omega}^T \circ A_{i^*} = A_{i^*}^\omega,$$

as ω is Hermitian. For a type-II matrix W in $\mathcal{A}^\omega$, the condition $WW^{(-)T} = nI$ reduces to

$$\sum_i \alpha_i A_i^\omega \cdot \sum_i \frac{1}{\alpha_i} A_{i^*}^\omega = nI.$$

We apply this to rank 2 and 3 weights in Sections 3 and 4 respectively.

3 Rank 2 weights

Let ω be a regular weight on K_n with $A_1 = J - I$ and $\mathcal{A}$ the 2-dimensional Bose-Mesner algebra of n by n matrices. The fact that $(I + A_1^\omega)^2$ is in the span of I and A_1^ω implies that the minimal polynomial of A_1^ω must be quadratic. We have

$$\begin{aligned}(A_1^\omega)^2 &= \beta_{11}^0 I + \beta_{11}^1 A_1^\omega \\ &= (n-1)I + \left(\beta_{11}^1(1) - \beta_{11}^1(-1)\right) A_1^\omega \\ &= (n-1)I + (n - 2a - 2)A_1^\omega.\end{aligned}$$

Thus, $\beta_{11}^1 = n - 2a - 2$. In Section 5 we will encounter 2-graphs along with regular weights on the lattice graphs. To avoid confusion, we will use C for the matrix of the 2-graph and write $C^2 = (n-1)I + AC$, where $A = n - 2a - 2$. Note that A is an integer—we will need this in Section 5.2.

3.1 Type-II matrices associated with rank 2 weights

Let C be the matrix of a regular 2-graph (n, a) as above. C has entries ± 1 off the diagonal and is symmetric, thus the matrix $I + \alpha C$ is type II if and only if

$$(I + \alpha C)(I + 1/\alpha C) = nI.$$

But this occurs when $\alpha + \dfrac{1}{\alpha} + A = 0$ which is equivalent to $\alpha^2 + A\alpha + 1 = 0$. Hence there are exactly two type-II matrices associated with the regular 2-graph when $A^2 \neq 4$, and exactly one otherwise. These type-II matrices are *generalized conference matrices*, defined in [5], and this result is a special case of Theorem 7.3 of [5].

In summary, type-II matrices associated with regular 2-graphs are given by

$$I + \alpha C \quad \text{where } \alpha = \frac{1}{2}\left(-A \pm \sqrt{A^2 - 4}\right).$$

Remark. When $A = 0$, we have the *conference 2-graphs.* The type-II matrices associated with a conference 2-graph are $I \pm iC$.

Example. The 2-graph with parameters $(n, A) = (16, -2)$ is unique. Using matrices given in [3] we construct the associated type-II matrix $W = I + C$. Here '+' represents 1 and '-' represents -1.

$$
\begin{pmatrix}
+ & - & - & - & - & - & - & + & + & + & + & + & + & + & + & + \\
- & + & - & - & + & + & + & - & - & - & + & + & + & + & + & + \\
- & - & + & - & + & + & + & + & + & + & - & - & - & + & + & + \\
- & - & - & + & + & + & + & + & + & + & + & + & + & - & - & - \\
- & + & + & + & + & - & - & - & + & + & - & + & + & - & + & + \\
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- & + & + & + & - & - & + & + & + & - & + & + & - & + & + & - \\
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+ & + & + & - & + & - & + & + & - & + & + & - & + & - & + & - \\
+ & + & + & - & + & + & - & + & + & - & + & + & - & - & - & +
\end{pmatrix}
$$

Computing $\mathcal{N}$ in *Maple*, we find $\dim(\mathcal{N}) = 16$, and W is a spin model. Regarding the discussion of switching in Section 2.6, let W' be the type-II matrix obtained by switching W on any vertex. Then W' is not a spin model, as is easily seen from the fact that the row sums are not constant and therefore $\mathbf{j}$ (the all ones vector) is not an eigenvector. The coherent closure of W is a rank 6 association scheme: $A_2^{\omega}(1)$ splits into two $(0, 1)$ matrices which, along with the remaining $A_i^{\omega}(\alpha)$, form the basic graphs of a non-p-polynomial scheme.

The type-II matrix W is a Hadamard matrix when $\alpha = \pm 1$, and this happens precisely when $A = \pm 2$. The Nomura algebras for these have been investigated in [10]. There exist Hadamard matrices of order 2^n whose Nomura algebras have dimension 2^n; the example above is an instance of this family with $n = 4$. In fact, the Nomura algebra is a product of 4 copies of the trivial Bose-Mesner algebra. When $n \geq 12$ and $n \equiv 4 \pmod 8$, the Nomura algebras are trivial.

Theorem 3.1. *Let W be a type-II matrix associated with a rank 2 regular weight. If W is not a Hadamard matrix, then $\mathcal{N}_W$ is trivial.*

Proof. Let C be a matrix of the 2-graph with $C_{1j} = 1$ for all j. (Use scaling to get C into this form.) Put $W = I + \alpha C$ as before, and consider the set $\{W_{i/1}\}$. These are eigenvectors for $\mathcal{N}_W$ by definition, and form a linearly independent set by (23) of [10]. For $i \neq j$ and $i, j \neq 1$, we make the claim: if $W_{i/1} \perp W_{j/1}$ then W is a Hadamard matrix. Indeed,

$$
\begin{aligned}
W_{i/1} \cdot W_{j/1} &= \sum_k W_{i/1}[k] W_{j/1}[k] \\
&= \frac{W_{ii}}{W_{i1}} \cdot \frac{W_{ij}}{W_{i1}} + \frac{W_{ji}}{W_{j1}} \cdot \frac{W_{jj}}{W_{j1}} + \sum_{k \neq i,j} \frac{W_{ki} W_{kj}}{W_{k1} W_{k1}} \\
&= \frac{2}{\alpha^2} W_{ij} + \alpha^2 C_{1i} C_{1j} + \sum_{k \neq 1,i,j} C_{ki} C_{kj} \\
&= \frac{2}{\alpha} C_{ij} + \alpha^2 - 1 + \sum_k C_{ki} C_{kj} \quad \text{since } C_{ii} = 0 \text{ and } C_{1i} = 1,
\end{aligned}
$$

$$= \frac{2}{\alpha} C_{ij} + \alpha^2 - 1 + AC_{ij} \quad \text{since the sum above is just the } i,j \text{ entry of } C^2,$$
$$= \left(\frac{2}{\alpha} + A\right) C_{ij} + \alpha^2 - 1.$$

All non-diagonal entries of C are ± 1. Hence $W_{i/1} \perp W_{j/1}$ only when $\frac{2}{\alpha} + A + \alpha^2 - 1 = 0$ or $-\frac{2}{\alpha} - A + \alpha^2 - 1 = 0$. Since α is a root of $x^2 + Ax + 1$, the first case implies $\frac{2}{\alpha} + A - A\alpha - 2 = 0$, which gives $\alpha = 1$ or $-\frac{2}{A}$. The second case implies $\frac{2}{\alpha} + A + A\alpha + 2 = 0$, giving $\alpha = -1$ or $-\frac{2}{A}$. If $\alpha = -\frac{2}{A}$ then $A = \pm 2$ and $\alpha = \pm 1$. Thus both cases reduce to $\alpha = \pm 1$ and W is a Hadamard matrix. This proves the claim.

When W is not Hadamard, the vectors $W_{i/1}$ for $i > 1$ must all belong to the same eigenspace of any matrix in $\mathcal{N}_W$. Hence $\mathcal{N}_W$ has only two eigenspaces, and is therefore of dimension 2.

□

Remark.

Combining this theorem with the results on Hadamard matrices above, we conclude that the only possibly interesting Nomura algebras associated with regular 2-graphs occur when $A = \pm 2$ and $n < 12$ or $n \equiv 0 \pmod 8$.

4 Rank 3 weights

For a given SRG Γ, suppose we have on hand a regular weight ω, with notation as in Section 2.4. The weighted intersection matrices are written

$$M_0^\omega = I, \quad M_1^\omega = \begin{pmatrix} & 1 & \\ k & A & B \\ & C & D \end{pmatrix}, \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & C & D \\ l & E & F \end{pmatrix}.$$

The parameters A–F are standing in for the corresponding β_{ij}^k's for convenience. It can be shown ([8]) that $C = Bl/k$ and $E = Dl/k$.

4.1 Type-II matrices in $\mathcal{A}^\omega$

Theorem 4.1. *The weighted Bose-Mesner algebra $\mathcal{A}^\omega$ of a regular weight with values in U_t on a strongly regular graph contains type-II matrices $I + \alpha A_1^\omega + \beta A_2^\omega$, where*

$$\alpha = \frac{1}{2}\left(X \pm \sqrt{X^2 - 4}\right), \quad \beta = \frac{1}{2}\left(Z \pm \sqrt{Z^2 - 4}\right),$$

with

$$X = -\frac{l}{k} BY - A - \frac{l}{k} D,$$
$$Z = -DY - B - F,$$

and Y *a root of the cubic:*

$$\frac{l}{k}BDY^3 + \left[\left(\frac{l}{k}-1\right)\left(D^2 - B^2\frac{l}{k}\right) + \frac{l}{k}BF + DA - 1\right]Y^2 + \\ \left[\left(1-\frac{2l}{k}\right)AB + AF + \left(\frac{l}{k}\left(1-\frac{2l}{k}\right)-2\right)BD + \left(\frac{l}{k}-2\right)DF\right]Y \\ - \left(A + \frac{l}{k}D\right)^2 - (B+F)^2 + 4 = 0.$$

Proof. Set $W = I + \alpha A_1^\omega + \beta A_2^\omega$ with α and β arbitrary complex numbers. Requiring that W be type II means $WW^{(-)T} = nI$. This gives

$$\begin{aligned} WW^{(-)T} &= (I + \alpha A_1^\omega + \beta A_2^\omega)\left(I + \frac{1}{\alpha}A_1^\omega + \frac{1}{\beta}A_2^\omega\right), \\ nI &= I + \left(\alpha + \frac{1}{\alpha}\right)A_1^\omega + \left(\beta + \frac{1}{\beta}\right)A_2^\omega + (A_1^\omega)^2 + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)A_1^\omega A_2^\omega + (A_2^\omega)^2. \end{aligned}$$

The products $(A_1^\omega)^2$, $A_1^\omega A_2^\omega = A_2^\omega A_1^\omega$ and $(A_2^\omega)^2$ may all be read from the intersection matrices above. We have:

$$\begin{aligned} (A_1^\omega)^2 &= kI + AA_1^\omega + BA_2^\omega, \\ A_1^\omega A_2^\omega &= \frac{l}{k}BA_1^\omega + DA_2^\omega, \\ (A_2^\omega)^2 &= lI + \frac{l}{k}DA_1^\omega + FA_2^\omega. \end{aligned}$$

Now, setting $X := \alpha + \frac{1}{\alpha}$, $\quad Y := \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, and $Z := \beta + \frac{1}{\beta}$,

$$nI = (1+k+l)I + \left(X + A + \frac{l}{k}BY + \frac{l}{k}D\right)A_1^\omega + (Z + B + DY + F)A_2^\omega,$$

which implies

$$X + A + \frac{l}{k}BY + \frac{l}{k}D = 0 \quad \text{and} \tag{1}$$

$$Z + B + DY + F = 0. \tag{2}$$

We see that

$$(2Y - XZ)^2 = (X^2-4)(Z^2-4) \tag{3}$$

and substituting into (3) for X and Z using (1) and (2) gives the required cubic in Y. □

Remarks.

1. Exactly one of the equations: $2Y = XZ \pm \sqrt{(X^2-4)(Z^2-4)}$ holds, and this determines which roots should be taken for α and β. Hence, the maximum number of distinct pairs (α, β) is 6. We shall see examples shortly that realize this maximum, and also cases in which only 2 distinct solutions are found.
2. If $\alpha, \beta \in \{\pm 1\}$, then W is a Hadamard matrix and the remarks in Section 3.1 apply.

5 $\mathbf{L_2(n)}$

The lattice graphs are rectangular grids with the lattice points as vertices. Two are adjacent if and only if they have exactly one coordinate in common. The SRG parameters, for $n \geq 2$, are $L_2(n) = \big(n^2, 2(n-1), n-2, 2\big)$. Note that when $n = 2$, the graph is strongly regular but *imprimitive*: the complement is not connected.

In this section we find all type-II matrices lying in the weighted Bose-Mesner algebra of a lattice graph. In this instance, we know precisely what the regular weights are—they are tensors of regular 2-graphs.

Theorem 5.1. ([15]) *If ω is a non-trivial regular weight with full support on the lattice graph $L_2(n)$ then n is even and $\omega = \omega_1 \otimes \omega_2$, where $\delta\omega_1$ and $\delta\omega_2$ are regular 2-graphs with the same parameters.*

Under a suitable ordering of the vertices, the adjacency matrices for the lattice graph have the form

$$\begin{aligned} A_1 &= I \otimes (J - I) + (J - I) \otimes I, \\ A_2 &= (J - I) \otimes (J - I). \end{aligned}$$

For the remainder of this section, we let ω be a regular weight on $L_2(n)$ which according to the theorem must have the form

$$\begin{aligned} \omega &= (I + C_1) \otimes (I + C_2) \\ &= I \otimes I + I \otimes C_2 + C_1 \otimes I + C_1 \otimes C_2, \end{aligned}$$

where C_i $(i = 1, 2)$ is a matrix of a regular 2-graph $\delta\omega_i$ with parameters (n, a).

5.1 Kronecker products

Since Kronecker products of type-II matrices are type II, we will always have type-II matrices in $\mathcal{A}^\omega$ that are products of type-II matrices related to the constituent 2-graphs. Explicitly, we observe that

$$W := (I + \alpha C_1) \otimes (I + \alpha C_2)$$

(with $\alpha^2 + A\alpha + 1 = 0$ as in Section 3.1) is type II. Expanding,

$$\begin{aligned} W &= I \otimes I + \alpha(I \otimes C_2 + C_1 \otimes I) + \alpha^2 (C_1 \otimes C_2) \\ &= I_{n^2} + \alpha A_1^\omega + \alpha^2 A_2^\omega. \end{aligned}$$

We conclude that $W \in \mathcal{A}^\omega$.

Note that there are two possibilities for α, giving two type-II matrices of this form, but $(I + \alpha_1 C_1) \otimes (I + \alpha_2 C_2)$ does not lie in $\mathcal{A}^\omega$ when $\alpha_1 \neq \alpha_2$.

We are now interested in finding all type-II matrices in $\mathcal{A}^\omega$, referring back to Theorem 4.1. (This discussion ends with Theorem 5.3, should the reader wish to get it over with.) From [15], the intersection matrices for $\mathcal{A}^\omega$ have the form:

$$M_1^\omega = \begin{pmatrix} & 1 & \\ 2(n-1) & A & 2 \\ & n-1 & 2A \end{pmatrix}, \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & n-1 & 2A \\ (n-1)^2 & (n-1)A & A^2 \end{pmatrix}.$$

Here, as in Section 3, the parameter A is related to the 2-graph parameters by $A = n - 2a - 2$. Replacing k, l, B, D, F with their counterparts from these, Theorem 4.1 gives

$$\alpha = \frac{1}{2}\left(X \pm \sqrt{X^2 - 4}\right), \quad \beta = \frac{1}{2}\left(Z \pm \sqrt{Z^2 - 4}\right),$$

with

$$X = -(n-1)Y - nA, \quad Z = -2AY - A^2 - 2 \tag{4}$$

and Y a root of

$$(Y + A)\left[2A(n-1)Y^2 + (A^2(n-3) - (n-2)^2)Y - A(n^2 + A^2 + 4)\right] = 0, \tag{5}$$

taking matching signs in the expressions for α and β when $2Y - XZ = -\sqrt{(X^2-4)(Z^2-4)}$ and opposite signs when $2Y - XZ = \sqrt{(X^2-4)(Z^2-4)}$.

Clearly $Y = -A$ is a solution. This gives $X = -A, Z = A^2 - 4$ and is precisely the case in which W is a tensor product. This is formalized in the following lemma.

Lemma 5.2. *Let Γ be the lattice graph $SRG(n^2, 2(n-1), n-2, 2)$ and suppose $W = I + \alpha A_1^\omega + \beta A_2^\omega$ is a type-II matrix in $\mathcal{A}^\omega$. The following are equivalent:*

1. $W = (I + \gamma C_1) \otimes (I + \gamma C_2)$ *for some* $\gamma \in \mathbb{C}$;
2. $\alpha = \gamma$, $\beta = \gamma^2$ *where* $\gamma^2 + A\gamma + 1 = 0$;
3. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = -A$.

Proof. 1⇒2: Expanding $(I + \gamma C_1) \otimes (I + \gamma C_2)$ to $I_{n^2} + \gamma(I \otimes C_2 + C_1 \otimes I) + \gamma^2 C_1 \otimes C_2$ gives $\alpha = \gamma$ and $\beta = \gamma^2$. Since $W^{(-)} = (I + \frac{1}{\gamma}C_1) \otimes (I + \frac{1}{\gamma}C_2)$,

$$\begin{aligned} WW^{(-)T} &= (I + \gamma C_1) \otimes (I + \gamma C_2) \ \cdot \ (I + 1/\gamma C_1) \otimes (I + 1/\gamma C_2) \\ &= (I + \gamma C_1)(I + 1/\gamma C_1) \ \otimes \ (I + \gamma C_2)(I + 1/\gamma C_2) \\ &= \left[I + (\gamma + 1/\gamma) C_1 + C_1^2\right] \ \otimes \ \left[I + (\gamma + 1/\gamma) C_2 + C_2^2\right]. \end{aligned}$$

Substituting $C_i^2 = (n-1)I + AC_i$,

$$\begin{aligned} WW^{(-)T} &= \left[nI + (\gamma + 1/\gamma + A) C_1\right] \ \otimes \ \left[nI + (\gamma + 1/\gamma + A) C_2\right] \\ &= n^2 I_{n^2} + n(\gamma + 1/\gamma + A)(I \otimes C_2 + C_1 \otimes I) + (\gamma + 1/\gamma + A)^2 C_1 \otimes C_2 \\ &= n^2 I_{n^2} + n(\gamma + 1/\gamma + A) A_1^\omega + (\gamma + 1/\gamma + A)^2 A_2^\omega. \end{aligned}$$

Since W is type II, this must equal $n^2 I_{n^2}$, thus $\gamma + 1/\gamma = -A$, and $\gamma^2 + A\gamma + 1 = 0$ follow from linear independence of A_1^ω and A_2^ω.

2⇒1: Immediate.

2⇒3: Given $\gamma^2 + A\gamma + 1 = 0$, the roots of $x^2 + Ax + 1$ are γ and $1/\gamma$. Assuming that $\alpha = \gamma$ and $\beta = \gamma^2$, we have

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{1}{\gamma} + \gamma = -A.$$

3⇒2: Suppose 3 holds, and solve for β:

$$\begin{aligned} \beta &= \frac{1}{2}\left(-A\alpha \pm \sqrt{A^2\alpha^2 - 4\alpha^2}\right) \\ &= \alpha\left(\frac{-A \pm \sqrt{A^2 - 4}}{2}\right), \end{aligned}$$

thus $\beta = \alpha\gamma$ where $\gamma^2 + A\gamma + 1 = 0$. It remains to show that $\alpha = \gamma$. Returning to (4) with $Y = -A$, we have $X = -A$. So $\alpha + 1/\alpha = -A$ and α is a root of $x^2 + Ax + 1$. Now $\alpha = \gamma$ or $\alpha = 1/\gamma$. In the latter case, $\beta = \alpha\gamma = 1$, so $Z = \beta + 1/\beta = 2$. But from (4), $Z = A^2 - 2$ forcing $A = \pm 2$. This implies $\gamma = \pm 1$ and thus $\gamma = 1/\gamma$. □

It is natural to ask when $\mathcal{A}$ and/or $\mathcal{A}^\omega$ are contained in $\mathcal{N}_W$. The following observation addresses this question for the lattice graph, when W is a tensor of type-II's. From Proposition 7 of [10], we have:

$$W = W_1 \otimes W_2 \Longrightarrow \mathcal{N}_W = \mathcal{N}_{W_1} \otimes \mathcal{N}_{W_2}.$$

We claim: If $W = U \otimes V$ with U and V $n \times n$ type-II matrices and $\mathcal{A}$ the Bose-Mesner algebra for a lattice graph $L_2(n)$, then $\mathcal{A} \subseteq \mathcal{N}_W$.

Proof. Since $N_W = N_U \otimes N_V$ and every Nomura algebra contains I and J, we have $I \otimes I$, $A_1 := I \otimes (J - I) + (J - I) \otimes I$, and $A_2 := (J - I) \otimes (J - I)$ all contained in $\mathcal{N}_W$. Under a suitable ordering of vertices, these three matrices form the standard basis for $\mathcal{A}$. □

5.2 Type-II matrices

To summarize, we have seen thus far that $\mathcal{A}^\omega$ for the lattice graph always contains type-II matrices which are tensors of type-II's related to the constituent 2-graphs of ω. The Nomura algebras are tensors of the corresponding Nomura algebras, and are spin models under the conditions of Theorem 2.2. We now show that $\mathcal{A}^\omega$ contains type-II matrices which are not tensors, as suggested by Lemma 5.2.

Theorem 5.3. *The weighted Bose-Mesner algebra of a regular weight of rank 3 on $L_2(n)$ ($n > 2$) contains precisely 2 type-II matrices when $A = 0$, 3 when $A = \pm 2$, and 6 otherwise. At most 2 of these are tensors.*

Proof. Consider the quadratic factor in (5), and label it Q. The discriminant of Q is

$$\left(A^2(n-3) - (n-2)^2\right)^2 + 8A^2(n-1)(n^2 + A^2 + 4)$$

which may be rewritten:

$$A^4(n+1)^2 + A^2(6(n^3 + n^2) - 8) + (n-2)^4$$

and is therefore positive for $n \geq 2$. Equation (5) thus has 3 distinct, real solutions unless $-A$ is a root of Q. Evaluating Q at $Y = -A$ gives $An(A^2 - 4)$. Hence the exceptional cases are $A = 0, \pm 2$.

Suppose now that $A \neq 0, \pm 2$. Since X and Z are both linear in Y, we have three distinct, real values for X and for Z respectively. As α satisfies $\alpha^2 - X\alpha + 1 = 0$, distinct values of X yield distinct values of α, and each X gives two solutions for α unless $X = \pm 2$. We conclude there are 6 distinct pairs (α, β) except when X and Z are both ± 2.

Suppose further that $X = \pm 2, Z = \pm 2$. In each of the four cases, we use (4) to equate two expressions for Y. The case $X = Z = 2$ gives

$$\frac{nA + 2}{-(n-1)} = \frac{A^2 + 4}{-2A} \Longrightarrow A = -2 \quad \text{or} \quad A = \frac{2(n-1)}{n+1}.$$

However, $A \neq -2$ by assumption, and clearly

$$0 < \frac{2(n-1)}{n+1} < 2.$$

Recalling that A is an integer, we have $A = 1$. But then $2(n-1) = n+1$ which gives $n = 3$, contradicting the fact that n is even by Theorem 5.1.

The case $X = 2, Z = -2$ gives

$$\frac{nA+2}{-(n-1)} = \frac{A^2}{-2A} \implies A = -\frac{4}{n+1}.$$

This requires $n = 1$ or $n = 3$, neither of which is possible.

The remaining two cases are similar, and lead to no solutions. □

We next analyse the exceptional cases to Theorem 5.3.

Case $A = 0$. (5) becomes $-(n-2)^2Y^2 = 0$, which implies $(X, Y, Z) = (0, 0, -2)$. We conclude that $\alpha = \pm i$, $\beta = -1$, and we have exactly two type-II matrices.

Case $A = 2$. Q becomes

$$4(n-1)Y^2 - (n-4)^2Y - 2(n^2+8),$$

so (5) has solutions $Y = -2$ (twice) and $Y = \frac{n^2+8}{4(n-1)}$. If $Y = -2$, $(X, Y, Z) = (-2, -2, 2)$, yielding $(\alpha, \beta) = (-1, 1)$. The third root gives

$$X = -\frac{n^2+8n+8}{4}, \quad Z = -\frac{n^2+6n+2}{n-1},$$

and we have two pairs (α, β), taking like signs for the radicals since $2Y - XZ > 0$.

Case $A = -2$. This case is identical to $A = 2$ except for signs.

The table below summarizes these exceptional cases.

Table 1. Exceptional cases for $L_2(n)$.

	X	Y	Z	(α, β)	W a tensor?
$A = 0$	0	0	-2	$(\pm i, -1)$	Y
$A = 2$	-2	-2	2	$(-1, 1)$	Y
	$-\frac{n^2+8n+8}{4}$	$\frac{n^2+8}{4(n-1)}$	$-\frac{n^2+6n+2}{n-1}$	2 solutions	N
$A = -2$	2	2	2	$(1, 1)$	Y
	$\frac{n^2+8n+8}{4}$	$-\frac{n^2+8}{4(n-1)}$	$-\frac{n^2+6n+2}{n-1}$	2 solutions	N

Remarks.

1. The case $A = 0$ gives the conference 2-graphs. Here, $\mathcal{N}$ is a product of $\mathrm{Span}(I, J)$ by Theorem 3.1 because the only type-II matrices are tensors of the form $I + iC_1 \otimes I + iC_2$, and the constituent type-II's are clearly not Hadamard matrices.
2. The 2-graphs with $A = -2$ and $A = 2$ are complements. Recall that the matrix C_i representing the 2-graph is taken as the $(0, \pm 1)$ adjacency matrix of a graph. Accordingly, replacing the 2-graph with its complement amounts to swapping $-C_i$ and C_i. Forming $\omega = (I - C_1) \otimes (I - C_2)$ we see that the entries of A_1^ω are negated while A_2^ω remains the same. Hence the type-II matrices associated with these complementary pairs of 2-graphs have $-\alpha$ and β as coefficients. The case $A = -2$, and the final two rows of the table, are therefore redundant.

3. The type-II's that are not tensors could be *generalized Kronecker products* as defined in [9]. The examples given in Tables 2–4 of the next section, by Lemma 4.3 of [9], are not generalized Kronecker products.

6 Examples

6.1 $L_2(n)$

As noted in Section 5, the Nomura algebra for a non-tensored type-II matrix may be non-trivial, even when the constituent 2-graphs produce only Potts models.

6.1.1 Lattice graphs from regular 2-graphs with $A = 0$ and $n \leq 50$

Nontrivial regular 2-graphs with $n \leq 50$ and $A = 0$ occur for $n = 10, 26$ and 50 ([3]). These all have trivial Nomura algebras by Remark 1 of Section 5.2.

6.1.2 Lattice graphs from regular 2-graphs with $A \neq 0$ and $n \leq 50$

Nontrivial regular 2-graphs with $A \neq 0$ and $n \leq 50$ occur for $n = 16, 28, 36$ ([3], [19]). In the first two cases, there is a unique regular 2-graph. For $n = 36$, 227 non-isomorphic regular 2-graphs are known.

The tables below show the 2-graph parameters and the coefficients α and β for type-II matrices associated with $L_2(n)$.

Table 2. Parameters for the regular 2-graph with $n = 16$.

n	A	α	β
16	2	-1	1
16	2	$-49 \pm 20\sqrt{6}$	$-\frac{59}{5} \pm \frac{24}{5}\sqrt{6}$

Like signs are taken for α and β in row 2, giving a total of 3 type-II's for $(n, A) = (16, 2)$. In the first row, W is a Hadamard matrix and $\mathcal{N}$ has dimension 16 but is the product of 4 copies of the trivial Nomura algebra ([10], [5]). (The example from Section 3.1 is an instance of this.) In the second row, $\mathcal{N}$ is found to be trivial.

Table 3. Parameters for the regular 2-graph with $n = 28$.

n	A	α	β
28	6	$-3 \pm 2\sqrt{2}$	$17 \mp 12\sqrt{2}$
28	6	$-\frac{238}{3} \pm \frac{5}{3}\sqrt{1009} \quad \pm \frac{2}{3}\sqrt{20465 \pm 595\sqrt{1009}}$	$-\frac{457}{27} \pm \frac{20}{27}\sqrt{1009} \quad \pm \frac{2}{27}\sqrt{152930 \pm 4570\sqrt{1009}}$

For row 1, we know by Theorem 3.1 that the Nomura algebras for the constituent 2-graphs are trivial. For the lattice graph, $\mathcal{N}$ is thus the product of Span(I, J). Row 2 represents the type-II's that are not tensors. Only 4 of the 64 possible $\pm$ combinations actually occur. If the reader will kindly allow a slightly cryptic presentation, these shall be expressed as the following ordered pairs where each integer should be interpreted as a binary numeral: $(3, 3)$, $(1, 1)$, $(6, 4)$, $(4, 6)$. For example, $(6, 4)$ is $(110, 100)$ which indicates $+ + -$ for α and $+ - -$ for β. We have a total of 6 type-II's for $(n, A) = (28, 6)$. Nomura algebras for row 2 have not been determined.

Table 4. Parameters for the regular 2-graphs with $n = 36$.

n	A	α	β
36	2	-1	1
36	2	$-199 \pm 60\sqrt{11}$	$-\frac{757}{35} \pm \sqrt{11}$

Row 1 represents a Hadamard matrix and since $n \equiv 4 \pmod 8$ we know by Theorem 3.1 that $\mathcal{N}$ is a product of $\mathrm{Span}(I, J)$. Row 2 shows a pair of type-II's that are not tensors. The best hope for a non-trivial Nomura algebra comes from pairing two non-isomorphic 2-graphs to form ω. Therefore there are $\binom{227}{2}$ cases to look at, of which only a few have been computed, revealing nothing of interest.

6.2 $S_4(q)$

A family of regular weights with values $\pm 1, \pm i$, constructed using the projective symplectic group $S_4(q)$, is described in [15].

Let Γ be $\mathrm{SRG}(q^3 + q^2 + q + 1, q(q + 1), q - 1, q + 1)$ from the rank 3 action of $S_4(q)$ on the totally isotropic lines of the symplectic geometry. For q an odd prime power, there is a regular weight of rank 3 on Γ with intersection matrices given by

$$M_1^\omega = \begin{pmatrix} 0 & 1 & 0 \\ q(q+1) & 0 & \pm(q+1) \\ 0 & \pm q^2 & 0 \end{pmatrix}, \qquad M_2^\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \pm q^2 & 0 \\ q^3 & 0 & \pm q(q-1) \end{pmatrix},$$

('+' if $q \equiv 1 \pmod 4$ and '−' if $q \equiv 3 \pmod 4$).

The type-II matrices in $\mathcal{A}^\omega$ are $I + \alpha A_1^\omega + \beta A_2^\omega$ where

$$\alpha^2 = \frac{-\beta(\beta q^2 + 1)}{\beta + q^2}, \quad \beta = \frac{-q^2 - 1 \pm \sqrt{(q^2+1)^2 - 4}}{2} \qquad \text{when } q \equiv 1 \pmod 4$$

and

$$\alpha^2 = \frac{\beta(\beta q^2 - 1)}{\beta - q^2}, \quad \beta = \frac{q^2 + 1 \pm \sqrt{(q^2+1)^2 - 4}}{2} \qquad \text{when } q \equiv 3 \pmod 4.$$

These have been explicitly constructed for $q = 3, 5$ and in both cases the Nomura algebra is $\mathrm{Span}(I, J)$.

6.3 $T(5)$

Let Γ be SRG $(10, 3, 0, 1)$ which is the well-known Petersen graph. A regular weight of rank 3 on Γ is determined by the action of the alternating group A_5 on the pairs from $\{1, 2, \ldots, n\}$ as described in [15]. This example appears also in [8] and [18]. Borrowing Seidel's construction, we set

$$A := \begin{pmatrix} 0&1&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{pmatrix}$$

and $B := J - I - A$. The adjacency matrices for the Petersen graph and $T(5)$ are given by

$$A_1 := \begin{pmatrix} A & I \\ I & B \end{pmatrix} \quad \text{and} \quad A_2 := J - I - A_1$$

respectively. The weighted adjacency matrices are

$$A_1^\omega = \begin{pmatrix} A & I \\ I & -B \end{pmatrix} \quad \text{and} \quad A_2^\omega = \begin{pmatrix} B & A-B \\ A-B & A \end{pmatrix}$$

with the type-II matrices in $\mathcal{A}^\omega$ given by

$$\alpha^2 = (3-4i)/5, \quad \beta = i \quad \text{and} \quad \alpha^2 = (3+4i)/5, \quad \beta = -i,$$

both of which yield trivial Nomura algebras.

This example is the smallest in a family of SRG's known as the *triangular graphs.* The vertices are the unordered pairs from an n-set. Two of these pairs are adjacent if and only if they have exactly one element in common. The SRG parameters are $T(n) = \left(\binom{n}{2}, 2(n-2), n-2, 4\right)$ for $n \geq 4$. The complement of the Petersen graph is $T(5)$.

Other examples of regular weights in the triangular graph family exist. In [16] these are viewed in the context of the Johnson scheme, so ranks higher than 3 are considered. These cases have not yet been explored for type-II matrices.

6.4 A_{2n}

The alternating group A_{2n} acts transitively on the pairs of disjoint n-sets, or *bisections.* This action has rank $\frac{n}{2}+1$ when n is even, and rank $\frac{n+1}{2}$ when n is odd, giving association schemes of the same ranks. The case $n = 5$ is of interest here, as this is when the association scheme has rank 3. The group action affords the SRG $\Gamma = (126, 25, 8, 4)$. A regular weight on Γ can be constructed with intersection matrices below.

$$M_1^\omega = \begin{pmatrix} & 1 & \\ 25 & 8 & 4 \\ & 16 & 3 \end{pmatrix} \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & 16 & 3 \\ 100 & 12 & 6 \end{pmatrix}$$

There are 6 type-II matrices, as in Theorem 4.1, two of them real-valued.

References

1. Bannai, E., Bannai, E., Jaeger, F.: On spin models, modular invariance, and duality. J. Algebraic Combin. **6**(3), 203–228 (1997)
2. Bannai, E., Ito, T.: Algebraic combinatorics. I: Association Schemes. The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA (1984)
3. Bussemaker, F.C., Mathon, R.A., Seidel, J.J.: Tables of two-graphs. In: Combinatorics and graph theory (Calcutta, 1980), *Lecture Notes in Math.*, vol. 885, pp. 70–112. Springer, Berlin (1981)
4. Cameron, P.J., van Lint, J.H.: Designs, Graphs, Codes and their Links, *London Math. Soc. Student Texts*, vol. 22. Cambridge University Press, Cambridge and New York (1991)
5. Chan, A., Godsil, C.: Type II matrices and combinatorial structures. arXiv0707.1836v1 [mathCO]
6. Godsil, C.D.: Algebraic combinatorics. Chapman and Hall Mathematics Series. Chapman & Hall, New York (1993)
7. Higman, D.G.: Coherent algebras. Linear Algebra Appl. **93**, 209–239 (1987)
8. Higman, D.G.: Weights and t-graphs. Bull. Soc. Math. Belg. Sér. A **42**(3), 501–521 (1990). Algebra, groups and geometry

9. Hosoya, R., Suzuki, H.: Type II matrices and their Bose-Mesner algebras. J. Algebraic Combin. **17**(1), 19–37 (2003)
10. Jaeger, F., Matsumoto, M., Nomura, K.: Bose-Mesner algebras related to type II matrices and spin models. J. Algebraic Combin. **8**(1), 39–72 (1998)
11. Jones, V.F.R.: On knot invariants related to some statistical mechanical models. Pacific J. Math. **137**(2), 311–334 (1989)
12. Klin, M., Rücker, C., Rücker, G., Tinhofer, G.: Algebraic combinatorics in mathematical chemistry, methods and algorithms i. permutation groups and coherent (cellular) algebras. MATCH Commun. Math. Comput. Chem. **40**, 7–138 (1999)
13. Nomura, K.: An algebra associated with a spin model. J. Algebraic Combin. **6**(1), 53–58 (1997)
14. Nomura, K.: Spin models and Bose-Mesner algebras. European J. Combin. **20**(7), 691–700 (1999)
15. Sankey, A.D.: Regular weights of full rank on strongly regular graphs. Israel J. Math. **95**, 1–23 (1996)
16. Sankey, A.D.: Regular weights and the Johnson scheme. Israel J. Math. **97**, 11–28 (1997)
17. Seidel, J.J.: A survey of two-graphs. In: Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo I, pp. 481–511. Atti dei Convegni Lincei, No. 17. Accad. Naz. Lincei, Rome (1976)
18. Seidel, J.J.: Polytopes and non-Euclidean geometry. Mitt. Math. Sem. Giessen **163**, 1–17 (1984)
19. Seidel, J.J.: More about two-graphs. In: Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity (Prachatice, 1990), *Ann. Discrete Math.*, vol. 51, pp. 297–308. North-Holland, Amsterdam (1992)
20. Seidel, J.J., Taylor, D.E.: Two-graphs, a second survey. In: Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), *Colloq. Math. Soc. János Bolyai*, vol. 25, pp. 689–711. North-Holland, Amsterdam (1981)
21. Taylor, D.E.: Regular 2-graphs. Proc. London Math. Soc. (3) **35**(2), 257–274 (1977)