REGULAR WEIGHTS AND THE JOHNSON SCHEME

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Abstract. This work addresses existence of weighted coherent configurations (cc's) in the special case of the Johnson schemes, $J(n,k)$. Many examples of weighted cc's occur in the context of a permutation representation of a group which affords the cc. We determine the ranks of the weighted Johnson schemes which arise in this "group case" in connection with the alternating group A_n. The association scheme $J(n,2)$ has rank 3, with the triangular graph $T(n)$ as one of its graphs. Conditions on feasible parameters for regular, full rank weights on $T(n)$ are established. We describe two infinite families of such feasible parameters.

0. Introduction

A "prototype for weighted coherent configurations" is the weighted triangular graph $T(5)$, realized in $\mathbb{R}^3$ by a regular icosahedron (J. J. Seidel, [9]). This example is of interest algebraically as it is associated with a monomial representation of the alternating group A_5 lying over a rank 3 permutation representation. It is discussed in this context by D. G. Higman in [5], where weighted coherent configurations (weighted cc's) are defined and the connection between a weighted cc and monomial representations of a group affording the cc is established. Many examples are presented there. Definitions relevant to the present paper are repeated below.

The $T(5)$ example can be described purely in terms of the geometry of the regular icosahedron ([6], [7]). There is a one-to-one correspondence between switching classes of weights and line systems of a certain type ([5]). In the $T(5)$ example, this yields a system of 10 lines in 3-space with two intersection angles which achieves both the special bound and the absolute bound (established by Delsarte, Goethals and Seidel in [3]) for the number of such lines ([8],[9]).

In [6], regular weights of full rank on strongly regular graphs, which are homogeneous, rank 3 cc's, are investigated. Definitions here are consistent with, but slightly more general than, those in [6]. In the present paper, we aim to generalize the $T(5)$ example in two ways. The monomial representation approach gives rise to many rank 2 regular weights on triangular graphs but no further rank 3

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examples. This is proven as a special case of a theorem giving the rank of the regular weight on the Johnson scheme $J(n,k)$ associated with the action of A_n on k-sets. The only cases in which the weight has full rank are $(n,k) = (5,2)$ (which is the $T(5)$ example) and $(n,k) = (6,3)$. Full rank regular weights on $T(n)$ not arising in this way are of course possible. Conditions on the parameters which can occur are given in section 2 and two infinite families of feasible parameters are found. No examples are known which realize these parameters for $n > 5$.

In section 1 we give preliminaries on association schemes, regular weights and 2-graphs. In section 2, feasible regular rank 3 weight parameters for $T(n)$ are given. The method outlined in [5] for constructing regular weights in the group case is described in section 3 for the specific case of weights with values which are fourth roots of unity and cc's which are association schemes. This method is applied in section 4 to the Johnson scheme, and a full-rank example on $J(6,3)$ is detailed in section 5.

1. Preliminaries

1.1. Association schemes. For a relation f on a finite set X, and $x \in X$, we write $f(x)$ for the set $\{y \in X |\ (x,y) \in f\}$.

Definition. An **association scheme** $\mathcal{X} = (X, f, I)$ is a finite set X with a set of nonempty, symmetric relations $(f_i)_{i \in I}$ such that

(1) $\{f_i\}_{i \in I}$ forms a partition of $X \times X$,
(2) $f_0 = diag(X \times X)$,
(3) For $(x,z) \in f_k,\ i,j,k \in I, p_{ij}^k(x,z) := |\ f_i(x) \cap f_j(z)\ |$ is independent of the choice of (x,z) in f_k.

We write $A_i,\ i \in I$ for the adjacency matrix of the relation f_i, and set $v_i = p_{ii}^0$, the valency of the i^{th} relation. The **adjacency algebra** is the linear span over $\mathbb{C}$ of $A_i, i \in I$. The **intersection numbers** for the adjacency algebra are the structure constants p_{ij}^k defined by

$$A_i \cdot A_j = \sum_{k \in I} p_{ij}^k A_k.$$

The **intersection matrices** are

$$M_j := \left(p_{ij}^k\right)_{i,k \in I}\ (j \in I)$$

and the map $A_j \mapsto M_j$ is an isomorphism of associative algebras.

Definition. A **strongly regular graph** (srg) is an association scheme with 3 relations.

The **parameters** of an srg are $(n, p_{11}^0, p_{11}^1, p_{11}^2)$, where $n = |X|$.

1.2. Regular weights. ([5])

Definition. Let M be a commutative, multiplicative monoid. A **weight with values in** M is a 2–cochain $\omega \in C^2(X, M)$.

Usually, we will take $M = U_t \cup \{0\}$, where U_t is the set of complex t^{th} roots of unity. In particular, we take $t = 2$ in the present paper except in the discussion of the Johnson scheme $\mathcal{J}(6, 3)$ where $t = 4$.

Since M may contain 0 we define σ: $M \to M$ by

$$\sigma(a) = \begin{cases} a^{-1} & a \in U_t \\ 0 & a = 0. \end{cases}$$

Observe that if $t = 2$, σ is the identity map on M. If ω is a weight,

$$\delta\omega(x, y, z) = \omega(y, z)\sigma(\omega(x, z))\omega(x, y)$$

defines a 3-cochain which assigns a weight to each triple or **triangle** (x, y, z).

Viewing ω as a matrix, the **switching class of** ω is the set of matrices obtained from ω by similarity transformation by the matrix $\mathrm{diag}(u(x_1), u(x_2), \dots , u(x_n))$ where u is a function from X to U_t.

Let $\mathcal{X} = (X, f, I)$ be an association scheme, $\omega \in C^2(X, M),\ i, j \in I,\ x, z \in X$, and $\alpha \in M$. Define

$$\beta^{\delta\omega}_{ij}(x, z, \alpha) := \big|\ \{y \in f_i(x) \cap f_j(z) \mid \delta\omega(x, y, z) = \alpha\}\ \big|.$$

We say $\delta\omega$ is **regular on** $\mathcal{X}$ if for fixed $(x, z) \in f_k,\ \beta^{\delta\omega}_{ij}(x, z, \alpha)$ is independent of the choice of $(x, z) \in f_k$. In this case we write $\beta^k_{ij}(\alpha) = \beta^{\delta\omega}_{ij}(x, z, \alpha)$.

We say $\omega \in C^2(X, M)$ is **defined** on the association scheme $\mathcal{X} = (X, f, I)$ if $f_i \cap supp(\omega) \neq \emptyset \implies f_i \subset supp(\omega)$. Define $I_\omega \subseteq I$ by $supp(\omega) = \cup_{i \in I_\omega} f_i$. Then the **rank** of ω on $\mathcal{X}$ is $rank_{\mathcal{X}}(\omega) = |I_\omega|$.

Definition. A weight ω is regular on the association scheme $\mathcal{X}$ if

(1) ω is defined on $\mathcal{X}$,
(2) $\delta\omega$ is regular on $\mathcal{X}$, and
(3) $\omega(x, z) = 0 \implies \sum \omega(x, y)\omega(y, z) = 0$,
where the sum is over $y \in f_i(x) \cap f_j(z)$, $(i, j \in I)$.

The **weighted adjacency matrices** are $A^\omega_i := \omega \circ A_i$ $(i \in I)$ (the Hadamard matrix product). The set $\{A^\omega_i | i \in I\}$ spans a self-adjoint subalgebra of the $n \times n$ matrices over $\mathbb{C}$, called the weighted adjacency algebra, $\mathcal{A}^\omega$. The structure constants for $\mathcal{A}^\omega$ are given by

$$A^\omega_i \cdot A^\omega_j = \sum_{i \in I_\omega} \beta^k_{ij} A^\omega_k$$

where $\beta_{ij}^k := \sum_{\alpha \in M} \alpha \beta_{ij}^k(\alpha)$. As with ordinary association schemes, the regular representation gives an isomorphism of associative algebras, with

$$A_j^\omega \mapsto M_j^\omega := \left(\beta_{ij}^k\right)_{i,k \in I_\omega} \qquad (j \in I_\omega).$$

If the weighted adjacency matrices commute, they can be simultaneously diagonalized. The eigenvalues are the same as those of the corresponding weighted intersection matrices, and the multiplicities are calculated using the traces of the matrices A_i^ω, $(A_i^\omega)^2$. Testing these multiplicities for integrality rules out some sets of potential weight parameters. We will call $\{\beta_{ij}^k\}_{i,j,k \in I_\omega}$ a set of **feasible weight parameters** for the association scheme if it cannot be ruled out in this way.

Properties. The properties listed below follow easily from the definitions, and are repeated from [5].

(1) $\beta_{ij}^0(\alpha) = \begin{cases} v_i & \text{if } i = j \text{ and } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$
(2) $\beta_{0j}^k(\alpha) = \begin{cases} 1 & \text{if } j = k \text{ and } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$
(3) $\beta_{ij}^k(\alpha) = \beta_{ji}^k(\sigma(\alpha))$
(4) $\beta_{ij}^k(\alpha) v_k = \beta_{kj}^i(\sigma(\alpha)) v_i = \beta_{ik}^j(\sigma(\alpha)) v_j$

A weight ω is **regular on a strongly regular graph** Γ if ω is regular on the association scheme whose non-trivial relations are given by Γ and $\overline{\Gamma}$. We observe that a regular weight with values ± 1 on the association scheme with 2 relations is equivalent to a regular 2–graph, defined below.

1.3. Regular 2-graphs. In the next section, we use the fact that a regular 2-graph requires an even number of vertices ([12]) to rule out some parameter sets.

Definition. A **2–graph** is a set of vertices X and a set of distinguished triples $\Delta \subseteq X^{(3)}$ with the property that every 4–subset of X contains an even number of triples in Δ.

Triples in Δ are called **coherent triples**.

A 2–graph may be viewed as a 3–coboundary with values ± 1. There is a switching class of graphs representing 2–cochains of which it is the boundary, so that a 2–graph is equivalent to a switching class of graphs. The adjacency matrix of a graph in this switching class is a matrix representative of the 2–graph ([10]).

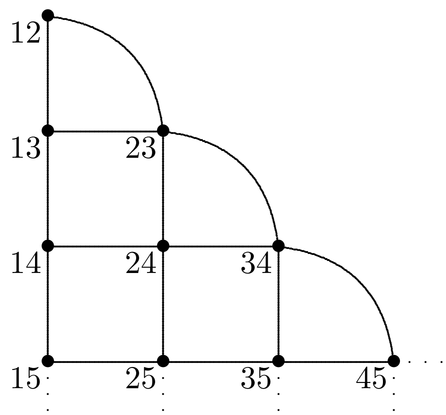
Definition. A 2–graph Φ is **regular** if and only if every pair of vertices is contained in the same number of coherent triples.

The **parameters** of a regular 2–graph are (n, a, b), where $n = 3a - 2b$ is the number of vertices, and a is the number of coherent triples containing a given pair. It is shown in [12] that both n and a are even.

We say a regular weight ω **fuses** to a regular 2–graph if the sum $A_1^\omega + A_2^\omega$ is a matrix of a regular 2–graph.

2. Feasible weight parameters for $T(n)$

2.1. The triangular graph T(n). Let $T(n)$ ($n \geq 5$) be the srg with unordered pairs from the set $\{1, 2, \ldots, n\}$ as vertices and two such pairs defined to be adjacent if and only if their intersection has cardinality 1.



In the figure above, each arc represents a clique of size $n-1$. The srg parameters are

$$\left(\binom{n}{2}, 2(n-2), n-2, 4\right).$$

The notation $T(n)$ will sometimes be used to refer to this parameter set rather than the graph itself. Intersection matrices for $T(n)$ are given below.

$$M_1 = \begin{pmatrix} & 1 & \\ 2(n-2) & n-2 & 4 \\ & n-3 & 2(n-4) \end{pmatrix}, \quad M_2 = \begin{pmatrix} & & 1 \\ & n-3 & 2(n-4) \\ \binom{n-2}{2} & \binom{n-3}{2} & \binom{n-4}{2} \end{pmatrix}.$$

2.2. Parameter restrictions. There is a (unique up to switching) regular rank 3 weight on the complement of $T(5)$, the so-called Petersen graph. This example is discussed in [5], [7] and [6]. For $n > 5$, there are two infinite families of feasible parameters, but no examples known to exist. The theorem below gives restrictions on the feasible parameter sets for $T(n)$.

Theorem. *If ω is a non-trivial regular weight with $M = \{\pm 1\}$ and full support on the triangular graph $T(n)$, then n is odd, and the weight parameters satisfy the following:*

(i) $\beta_{11}^2 = 0$

(ii) ${\beta_{11}^1}^2 + 8(n-2)$ *is a square and* $\beta_{21}^2 = \frac{1}{2}\left(\beta_{11}^1 \pm \sqrt{{\beta_{11}^1}^2 + 8(n-2)}\right)$

(iii) *Either* $\beta_{22}^2 = 0$, *in which case* $n \equiv 1 \pmod 4$, *or* ${\beta_{22}^2}^2 + (n-3)({\beta_{21}^2}^2 + 2(n-2))$ *is a square*

(iv) m_1, m_2, *and* m_3 *are positive integers, where*

$$m_1 = \frac{n(n-1)(n-2)(n-3)}{2K(K+\beta_{22}^2)}$$

$$m_2 = \frac{n(n-1)(n-2)(n-3)}{2K(K-\beta_{22}^2)},$$

$$m_3 = \frac{n(n-1){\beta_{21}^2}^2}{2({\beta_{21}^2}^2 + 2(n-2))}$$

and $K = \sqrt{{\beta_{22}^2}^2 + (n-3)({\beta_{21}^2}^2 + 2(n-2))}$.

Proof of (i). Let $\Gamma = T(n)$ and suppose ω is as in the statement of the theorem. Let $\beta_{ij}^k(\alpha)$ and β_{ij}^k be the usual weight parameters, so by the properties in section 2,

$$\beta_{11}^1 = 2\beta_{11}^1(1) - (n-2) \qquad \beta_{21}^2 = 2\beta_{21}^2(1) - 2(n-4)$$

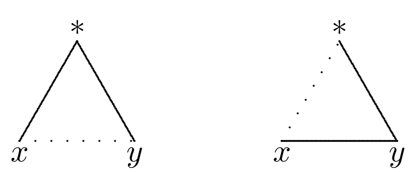
$$\beta_{11}^2 = 2\beta_{11}^2(1) - 4 \qquad \beta_{22}^1 = \beta_{21}^2(n-3)/4$$

$$\beta_{21}^1 = \beta_{11}^2(n-3)/4 \qquad \beta_{22}^2 = 2\beta_{22}^2 - \binom{n-4}{2}$$

and $\beta_{ij}^k(\alpha)$ is nonnegative and bounded by p_{ij}^k. Replacing A_i^ω by $-A_i^\omega$ if necessary, we may assume $\beta_{11}^2(1) \in \{0, 1, 2\}$. We may also assume that $\omega(12, xy) = +1$ for all pairs $\{x, y\}$. We abuse notation here, writing xy for $\{x, y\}$.

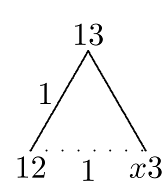
We show $\beta_{11}^2(1) \neq 0$, $\beta_{11}^2(1) \neq 1$.

Suppose $\beta_{11}^2(1) = 0$, so $\beta_{11}^2 = -4$ and $\beta_{21}^1 = 3 - n$. Then all triples $(x, *, y)$ of types $(1, 1, 2)$ and $(2, 1, 1)$



have weight -1.

Consider triples $(13, *, 23)$ of type $(1, 1, 1)$. These are $(13, 12, 23)$ and, for all $x > 3$, $(13, x3, 23)$. From



$(12, 13, x3)$ of type $(1, 1, 2)$ we have

$$\omega(13, x3) = -1.$$

Likewise, the triple $(12, 23, x3)$ forces

$$\omega(23, x3) = -1.$$

Setting $\alpha = \omega(13, 23)$, we find all triples $(13, *, 23)$ of type $(1, 1, 1)$ have weight α. It follows that

$$\beta^1_{11}(\alpha) = n - 2.$$

Replacing 3 by 4 in the above argument and taking $x = 3$ or $x > 4$, we have

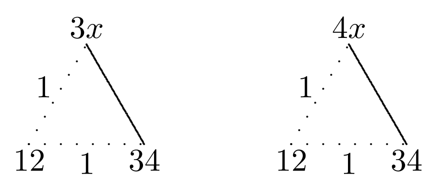
$$\omega(14, x4) = \omega(24, x4) = -1.$$

Also,

$$\omega(14, 24) = \alpha$$

since the triple $(14, 12, 24)$ has weight α. Next we apply this to triples $(12, *, 34)$ of type $(2, 1, 2)$. These are $(12, 3x, 34)$ and $(12, 4x, 34)$ for all $x > 4$. We claim

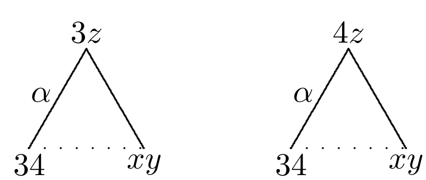
$$\omega(34, 3x) = \omega(34, 4x) = \alpha.$$



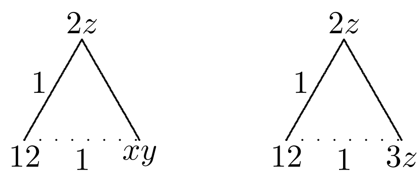
Indeed, this follows from the fact that triples $(13, 3x, 34)$ and $(14, 4x, 34)$ $(x > 4)$ have type $(1, 1, 1)$ and therefore weight α. Now, triples $(12, 3x, 34)$ and $(12, 4x, 34)$ have weight α. Thus,

$$\beta^2_{21}(\alpha) = 2(n - 4).$$

Finally, we consider triples $(12, xy, 34)$ with $(x, y > 4)$ having type $(2, 2, 2)$. We will show $\omega(34, xy) = -1$ to conclude that all such triples have weight -1. Triples $(34, *, xy)$ of type $(1, 1, 2)$ have weight -1.



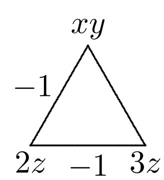
These are $(34, 3z, xy)$ and $(34, 4z, xy)$ with $z = x$ or $z = y$. It is enough to show $\omega(xy, *) = \alpha$, since $\omega(34, *) = \alpha$ from above. Triples $(12, 2z, xy)$ and $(12, 2z, 3z)$ $(z = x, y)$ of type $(1, 1, 2)$ have weight -1.



Hence,

$$\omega(2z, xy) = \omega(2z, 3z) = -1.$$

Now $(2z, xy, 3z)$ has type $(1, 1, 1)$,



thus $\omega(xy, 3z) = \alpha$. Similarly, $\omega(xy, 4z) = \alpha$ and we conclude

$$\beta^2_{22}(-1) = \binom{n-4}{2}.$$

We have

$$\omega = I + \alpha A_1 - A_2,$$

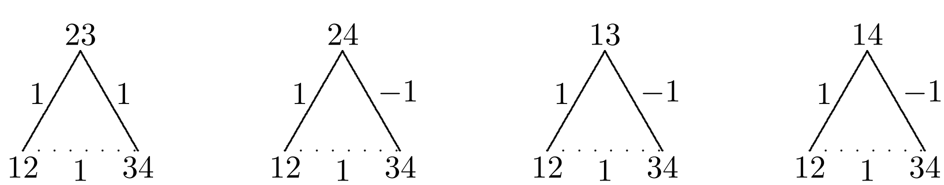
thus $\beta^2_{11}(1) = 0$ implies ω is trivial.

Suppose $\beta^2_{11}(1) = 1$. Then

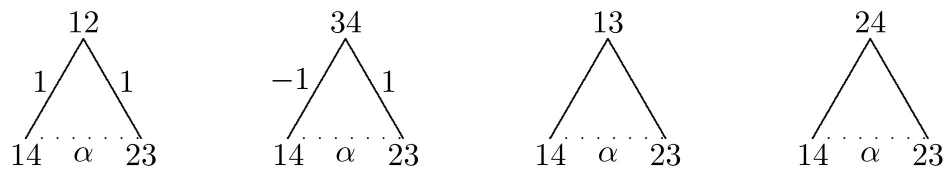
$$\beta^2_{11} = -2, \quad \beta^1_{21} = \frac{-(n-2)}{2}.$$

There are four triples $(12, *, 34)$ of type $(1, 1, 2)$, exactly one of which has weight 1. We may suppose that $(12, 23, 34)$ has weight 1, so $\omega(23, 34) = 1$. It follows that

$$\omega(24, 34) = \omega(13, 34) = \omega(14, 34) = -1.$$



Consider triples $(14,*,23)$ of type $(1,1,2)$. Setting $\alpha=\omega(14,23)$, we find $(14,12,23)$ and $(14,34,23)$ have weight α and $-\alpha$ respectively. Clearly, one of these is $+1$, hence the remaining two triples $(14,*,23)$ of type $(1,1,2)$ have weight -1.



This implies

$$\omega(13,14)=-\alpha\omega(13,23)$$

and

$$\omega(14,24)=-\alpha\omega(23,24).$$

Finally, consider triples $(13,*,24)$ of type $(1,1,2)$. Let $\beta=\omega(13,24)$. Triples $(13,12,24)$ and $(13,34,24)$ both have weight β, which implies $\beta=-1$ and that $(13,14,24)$, $(13,23,24)$ have different weights. But

$$\begin{aligned}\delta\omega(13,14,24)&=\omega(13,14)\omega(13,24)\omega(14,24)\\&=-\alpha\omega(13,23)\omega(13,24)(-\alpha)\omega(23,24)\\&=\delta\omega(13,23,24),\end{aligned}$$

a contradiction. Therefore, $\beta^2_{11}(1)\neq 1$. $\square$

Proofs of (ii)–(iv). To prove (ii)—(iv) we analyse the eigenvalues and their multiplicities for A^ω_1 and A^ω_2. Set $S=\sqrt{{\beta^1_{11}}^2+8(n-2)}$. Then A^ω_1 has eigenvalues

$$\beta^2_{21},\ \frac{\beta^1_{11}+S}{2},\ \frac{\beta^1_{11}-S}{2}$$

and A^ω_2 has eigenvalues

$$0,\ \frac{\beta^2_{22}+K}{2},\ \frac{\beta^2_{22}-K}{2}.$$

We need to determine the pairs of eigenvalues which share the same multiplicities. From the weighted intersection matrices, we have the equations

$$A^\omega_1A^\omega_2=\beta^2_{21}A^\omega_2 \tag{1}$$

$$(A^\omega_2)^2=\binom{n-2}{2}I+\beta^1_{22}A^\omega_1+\beta^2_{22}A^\omega_2. \tag{2}$$

Equation (1) implies that all non-zero eigenvalues of A_2^ω correspond to β_{21}^2. From (2), we have

$$\begin{aligned}
\frac{\beta_{21}^2(n-3)}{4}\frac{(\beta_{11}^1 \pm S)}{2} &= -\frac{(n-2)(n-3)}{2} \\
\implies \beta_{21}^2(\beta_{11}^1 \pm S) &= -4(n-2) \\
\implies \beta_{21}^2(\beta_{11}^1 \pm S) &= \frac{({\beta_{11}^1}^2 - S^2)}{2}.
\end{aligned}$$

So $\beta_{21}^2 = (\beta_{11}^1 \pm S)/2$, the sign depending on which one is even. This proves (ii).

Let m_1, m_2, m_3 be the multiplicities of $(\beta_{22}^2 + K)/2$, $(\beta_{22}^2 - K)/2$, and 0 respectively. Using the trace of A_i^ω and the trace of $(A_i^\omega)^2$, we have the five equations:

$$\begin{aligned}
m_1 + m_2 + m_3 &= \frac{n(n-1)}{2} \\
m_1\beta_{21}^2 + m_2\beta_{21}^2 + m_3(\beta_{11}^1 - \beta_{21}^2) &= 0 \\
m_1{\beta_{21}^2}^2 + m_2{\beta_{21}^2}^2 + m_3(\beta_{11}^1 - \beta_{21}^2)^2 &= n(n-1)(n-2) \\
m_1\Big(\frac{\beta_{22}^2 + K}{2}\Big) + m_2\Big(\frac{\beta_{22}^2 - K}{2}\Big) &= 0 \\
m_1\Big(\frac{\beta_{22}^2 + K}{2}\Big)^2 + m_2\Big(\frac{\beta_{22}^2 - K}{2}\Big)^2 &= \frac{n(n-1)(n-2)(n-3)}{4}.
\end{aligned}$$

Observe that if K is not rational, then $m_1 = m_2$, and it follows that $\beta_{22}^2 = 0$. But

$$\beta_{22}^2 = 2\beta_{22}^2(1) - \frac{(n-4)(n-5)}{2} \implies n \equiv 1 \pmod 4,$$

proving (iii).

Solving the equations above, we find

$$\begin{aligned}
m_1 &= \frac{n(n-1)(n-2)(n-3)}{2K(K+\beta_{22}^2)} \\
m_2 &= \frac{n(n-1)(n-2)(n-3)}{2K(K-\beta_{22}^2)} \\
m_3 &= \frac{n(n-1){\beta_{21}^2}^2}{2({\beta_{21}^2}^2 + 2(n-2))}
\end{aligned}$$

which proves (iv). □

2.3. Feasible parameter families. For any odd n, we find $\beta_{11}^1 = n-4$ and $\beta_{21}^2 = -2$ satisfy (ii), but there is not always a corresponding value of β_{22}^2 which satisfies (iii) and (iv). However, there are two families of parameters which occur for certain n.

	β_{11}^1	β_{21}^2	β_{22}^2	m_1	m_2	m_3
$n \equiv 1 \pmod 4$	$n-4$	-2	0	$\frac{(n-1)(n-2)}{4}$	$\frac{(n-1)(n-2)}{4}$	$n-1$
$n \equiv 3, 5 \pmod 6$	$n-4$	-2	$\pm\frac{n+3}{2}$	$\frac{n(n-2)}{3}$	$\frac{(n-2)(n-3)}{6}$	$n-1$

In the second case above, there is fusion to a regular 2–graph, which means that the parameters are ruled out when $\binom{n}{2}$ is odd. This occurs $\iff n \equiv 3 \pmod{12}$ or $n \equiv 11 \pmod{12}$.

For $n = 5$, the only feasible parameters are those from the first family above, and these match the known $T(5)$ example.

n	β_{11}^1	β_{21}^2	β_{22}^2	m_1	m_2	m_3
5	1	-2	0	3	3	4

For $n = 7$, K must be rational, but there are no possible values for β_{22}^2, thus there are no feasible parameters.

For $n = 9$, there are exactly 2 sets of feasible parameters, one from each of the two families above.

n	β_{11}^1	β_{21}^2	β_{22}^2	m_1	m_2	m_3
9	5	-2	0	14	14	8
9	5	-2	± 6	7	21	8

In the second case, we have fusion to a regular 2–graph on 36 points, and there are many which share these parameters (see [1]).

For $n = 11$, there is one set of feasible parameters. This is from the second family above, and is ruled out by fusion.

There are feasible parameter sets not belonging to either of these two families. The smallest occurrence is given below.

n	β_{11}^1	β_{21}^2	β_{22}^2	m_1	m_2	m_3
27	5	10	25	36	81	234

3. The group case

Let G be a finite group acting transitively on a set X with symmetric orbitals. Then the orbitals are the basic relations for an association scheme ([4] and others). It is well known that the centralizer algebra of the matrices in the permutation representation of G is the adjacency algebra of the association scheme. D. G. Higman ([5]) gave a generalization to transitive monomial representations. The centralizer algebra in this case is a weighted adjacency algebra and the weight is regular on the cc afforded by the underlying action of the group. We introduce notation and conventions below for the special case of monomial representations with associated weights having values in U_4 on the association scheme. The reader is directed to [5] for the full generality of the subject.

We fix $x \in X$ and suppose the stabilizer, $H := G_x$, has an index 2 subgroup A. Define λ to be the alternating character of H with respect to A:

$$\lambda(h) = \begin{cases} 1 & \text{if } h \in A \\ -1 & \text{otherwise.} \end{cases}$$

The induced monomial representation $\Gamma := \lambda^G$ has centralizer algebra spanned by weighted adjacency matrices. It may be that the associated weight ω is trivial or has rank smaller than the rank of the scheme. The entries in the weighted adjacency matrices are 0 and fourth roots of unity, thus it may happen that ω has non-real values.

To determine the rank of ω, we fix a transversal $\{t_1 = 1, t_2, \ldots, t_n\}$ to H in G. The action of G on X is equivalent to the action on cosets modulo H, so we may label the points as cosets t_iH. The nonzero entry in column j of $\Gamma(\sigma)$ for $\sigma \in G$ is then

$$\lambda(t_i^{-1}\sigma t_j)$$

in row i, where $\sigma t_j \in t_iH$. The entries in A_i^ω are determined by

$$A_i^\omega(x,y) = \begin{cases} \alpha_i \lambda(t_k^{-1}\sigma) \cdot \lambda(t_j^{-1}\sigma t_i) & \text{if } \sigma(H, t_iH) = (x,y) \\ 0 & \text{if } (x,y) \notin f_i \end{cases}$$

where $\sigma \in t_kH$, $\sigma t_i \in t_jH$ and $\alpha_i = 1$ if ω is real-valued. Specifically, α_i is determined by choosing $\tau_i \in G$ such that

$$\tau_i(H, t_iH) = (t_iH, H)$$

which must be possible since A_i^ω is an Hermitian matrix. Then

$$\alpha_i \lambda(t_i^{-1}\tau_i)\lambda(\tau_i t_i) = \overline{\alpha_i}$$

hence

$$\alpha_i^2 = \lambda(t_i^{-1}\tau_i^2 t_i).$$

The **relevant orbitals** are those indexed by i such that $A_i^\omega(x,y)$ is well-defined. That is, all i such that

$$\lambda(\sigma) = \lambda(t_i^{-1}\sigma t_i) \quad \forall \sigma \in G_{H,t_iH}.$$

The weight ω then has rank equal to the number of relevant orbitals.

The $T(5)$ example arises in this way, afforded by the action of the alternating group A_5 on pairs from $\{1,2,3,4,5\}$. Regular weights on $T(n)$ afforded by the action of A_n for $n > 5$ have rank 2, thus do not supply examples to realize the feasible rank 3 parameters of section 2. The proof of this fact is given in the next section, where we discuss the group approach in the more general setting of the Johnson scheme.

4. The Johnson scheme

Let $\mathcal{J} = J(n,k)$ be the well-known Johnson scheme defined as follows. Let $X = \{1,2,\ldots,n\}$ with $n \geq 5$. Fix $k \leq \dfrac{n}{2}$ and make the following relations on the $\dbinom{n}{k}$ k-element subsets of X: two sets are j^{th} associates if and only if their intersection has cardinality $k-j$. Observe that $\mathcal{J}$ is a rank $k+1$ association scheme, equivalent to $T(n)$ when $k=2$.

The group $G = A_n$ acts rank $k+1$ on the k-sets, affording the Johnson scheme. The stabilizer of the k-set $x = \{1,2,\ldots,k\}$ is

$$G_x \simeq (A_k \times A_{n-k})\,.\langle(1,2)(k+1,k+2)\rangle.$$

Since G_x contains an index 2 subgroup $A \simeq A_k \times A_{n-k}$, there is a regular weight on $\mathcal{J}$ associated with a monomial representation of G. The theorem below gives the relevant orbitals for the weighted scheme.

Theorem. *Let ω be a regular weight on $\mathcal{J}$ such that the weighted adjacency algebra is the centralizer algebra of a monomial representation of $G = A_n$ induced from the linear representation of G_x with kernel $A \simeq A_k \times A_{n-k}$. Then the indices of the relevant orbitals of the weighted scheme are given by the table below.*

$k < \lfloor \frac{n}{2} \rfloor$	$k = \frac{n-1}{2}$ (n odd)	$k = \frac{n}{2}$ (n even)
$0,1$	$0,1,k$	$0,1,k-1,k$

Remark. The only cases in which ω has full rank are $(n,k) = (5,2)$ and $(n,k) = (6,3)$.

Proof. Let $G = A_n$ and define $x,\ G_x,\ A,\ \lambda$ as above. Let Γ be the monomial representation of G induced from λ and let I_ω be the indexing set for the relevant orbitals of the associated weighted scheme $\mathcal{J}^\omega$.

Choose $t_j \in G$ as follows so that

$$t_j x := \{1,2,\ldots,k-j,k+1,k+2,\ldots,k+j\} \qquad (1 \leq j \leq k)$$

is a representative for the j^{th} orbit under G_x:

$$\begin{aligned} t_j &= (k-j+1,k+1)(k-j+2,k+2)\cdots(k,k+j) \quad \text{for } j \text{ even},\\ t_j &= (k-j+1,k+1)(k-j+2,k+2)\cdots(k,k+j)(k+1,k+2) \quad j>1 \text{ odd},\\ t_1 &= (k,k+1)(k+2,k+3). \end{aligned}$$

Recall

$$j \in I_\omega \iff \lambda(t_j^{-1}ht_j) = \lambda(h) \qquad \forall h \in G_x \cap G_{t_jx}.$$

For $h \in G_x$, $\lambda(h) = 1$ if and only if $h \in A$, that is if h acts evenly on x.

We show $1 \in I_\omega$. For $h \in G_x \cap G_{t_1x}$, h fixes k and $k+1$ pointwise and fixes the set $\{1, 2, \dots, k-1\}$. So

$$t_1^{-1}ht_1 = (k+2,k+3)h(k+2,k+3)$$

which acts evenly on x if and only if h does. Thus $1 \in I_\omega$.

Next, we show $j \notin I_\omega$ for $j > 1,\ k+j+2 \le n$. Take

$$h = (k+j-1,k+j)(k+j+1,k+j+2) \in G_x \cap G_{t_jx}.$$

Then

$$t_j^{-1}ht_j = (k-1,k)(k+j+1,k+j+2)$$

which acts as the transposition $(k-1,k)$ on x. But h fixes x pointwise, thus $j \notin I_\omega$.

It remains to treat the cases $k+j+2 > n$. Pairs (k,j) which occur here are

$$\left(\frac{n-1}{2}, \frac{n-1}{2}\right), \quad \left(\frac{n}{2}, \frac{n}{2}-1\right) \quad \text{and} \quad \left(\frac{n}{2}, \frac{n}{2}\right).$$

Suppose $k \ge \dfrac{n-1}{2}$, $\quad j = k$. Since t_j takes $i \in x$ to $i+k$, $t_j^{-1}ht_j$ acts on x just as h acts on the set $\{k+1, k+2, \dots, 2k\}$. But h acts evenly on this set if and only if h acts evenly on x. Thus $j \in I_\omega$.

Suppose $k = \dfrac{n}{2}$ and $j = k-1$. For $h \in G_x \cap G_{t_jx}$, h fixes the sets

$$\{1\}, \quad \{2,3,\dots,k\}, \quad \{k+1,k+2,\dots,n-1\}, \quad \{n\}.$$

Therefore h must have the same parity on x and $\{k+1, k+2, \dots, n\}$. Since t_j translates $i \in x,\ i > 1$ by $k-1$, $t_j^{-1}ht_j$ acts on $\{2, 3, \dots, k\}$ just as h acts on $\{k+1, k+2, \dots, n-1\}$. Hence $t_j^{-1}ht_j$ has the same parity on x as h, and $j \in I_\omega$. $\square$

5. Example: $\mathcal{J}(6,3)$

We use the notation of section 4, where $n = 6$, $k = 3$, so $G = A_6$.

5.1. Intersection matrices. Intersection matrices for the $\mathcal{J}(6,3)$ scheme are

$$M_0 = I_4, \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9 & 4 & 4 & 0 \\ 0 & 4 & 4 & 9 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 4 & 4 & 9 \\ 9 & 4 & 4 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

5.2. Weighted intersection matrices. We compute α_i as in section 3 to see that ω is not real-valued. We have

$$\begin{aligned} t_1 &= (34)(56) \\ t_2 &= (24)(35) \\ t_3 &= (1425)(36). \end{aligned}$$

In each case, we may take $\tau_i = t_i$. For $i = 1, 2$ t_i is a transposition and $\alpha_i^2 = \lambda(1) = 1$. But

$$\alpha_3^2 = \lambda(t_3^2) = \lambda\left((12)(45)\right) = -1$$

so entries of A_3^ω are $\pm i$ and ω is not real-valued.

The weighted intersection matrices are given below.

$$M_0^\omega = I_4, \qquad M_1^\omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9 & 0 & 2 & 0 \\ 0 & 2 & -2 & -9i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$M_2^\omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 9i \\ 9 & -2 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} & & & 1 \\ & & -i & \\ & i & & \\ 1 & & & \end{pmatrix}$$

Remark. It is clear from the matrices above that the weighted adjacency algebra $\mathcal{A}^\omega$ is not commutative. Hence, the trace character ζ is not the sum of linear constituents ([4]). Degrees (multiplicities) of the constituents of ζ correspond to multiplicities (degrees) of the constituents of the monomial representation Γ of G. Thus Γ has irreducible constituents of multiplicity greater than 1. Let

$$\zeta = \sum_{i=1}^{r} z_i \zeta_i$$

where ζ_i is irreducible of degree e_i. Since $\mathcal{A}^\omega$ has dimension 4,

$$\sum_{i=1}^{r} e_i^2 = 4$$

([4, section 5]). We conclude that $e_1 = 2$ and $r = 1$. So, ζ has one degree 2 constituent which must have multiplicity 10. It follows that Γ has a single degree 10 constituent of multiplicity 2. In the character table for G in [2], this is χ_7.

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