QUOTIENTS OF COHERENT CONFIGURATIONS

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Abstract. A result of C. D. Godsil and W. J. Martin ([GM]) gives conditions under which a partition of the vertex set of an association scheme induces a quotient association scheme. This work extends that result to coherent configurations. We provide a concrete description of the basic relations of the quotient scheme, and a characterization of its parameters. Equivalent conditions to those in [GM] are determined.

1. Introduction

The standard approach to quotient association schemes is to find, within an imprimitive association scheme, a subset of the basic relations or colors which forms an equivalence relation. This determines a partition of the vertex set, and also induces a partition of the colors. The quotient scheme then has equivalence classes of vertices as points, and equivalence classes of colors as its basic relations ([BCN], [BI]). This technique is also applied in the more general settings of coherent configurations, semi-coherent configurations, and relation schemes ([DGH1], [DGH2]).

Quotients in the sense of Godsil and Martin ([GM]) are not necessarily derived from equivalence relations on colors. That is, examples exist in which two points in the same cell of a vertex partition are i-related, and two points in different cells are i-related as well. What distinguishes the basic relations in the quotient scheme, then, is not a particular subset or equivalence class of colors, but rather a multiset. For example, cells $[x]$ and $[y]$ may be joined by 3 i-arcs, 5 j-arcs and 2 k-arcs, while $[x]$ is joined to $[z]$ by 3 i's, 4 j's and 3 k's.

The supposition of an *equitable* partition σ of the vertex set is the condition that ensures these multisets are well-defined: the multiset of colors from $x_1 \in [x]$ to $[y]$ does not depend on the choice of x_1. In [GM], further assumptions that i) σ has *pairwise isometric* cells; and ii) cells of σ are *simple* imply that the quotient algebra modulo σ is the Bose-Mesner algebra of an association scheme. We show the multisets are the basic relations for this quotient scheme, and extend this to the setting of coherent configurations.

In section 2 we give some necessary definitions and notational conventions. The multisets are defined in section 3, and the isometric and simple properties are discussed in this context. Section 4 includes equivalent conditions to those in [GM], under which a quotient is coherent. The parameters of such a quotient scheme are described in section 5.

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2. Preliminaries

For basic definitions and facts about coherent configurations, the reader is referred to [DGH1]. One version of the defining axioms is given below for reference. Let $\mathcal{A}$ be a coherent configuration (cc) with vertex set X and basic relations given by matrices $A_0, A_1, \ldots, A_d$. These $(0,1)$ matrices satisfy:

2.1.

(i) $\sum_{i\in\Omega} A_i = I$ for some $\Omega \subseteq \{0,\ldots,d\}$,

(ii) $\sum_{i=0}^{d} A_i = J$,

(iii) $A_i^T = A_{i^*}$, $i^* \in \{0,\ldots,d\}$

(iv) $A_i \cdot A_j = \sum_{i=0}^{d} p_{ij}^k A_k$.

The coherent algebra $\langle A_i\rangle_{i=0,\ldots,d}$, where the span is over $\mathbb{C}$, will be denoted $\mathbb{A}$. The term *color* will be used interchangeably with *relation*. We will say the color i *joins* a vertex x to a vertex y if x is i-related to y.

Condition 2.1 (i) determines the *standard partition* Σ of the vertices into types: $x \in X$ has *type* i if x is i-related to itself.

Let σ be a partition of the vertex set X. Write $[x]$ for the cell of σ containing x. We say σ is *equitable* if for any ordered pair of cells $([x],[y])$ and for any color i, the number of i-arcs starting at $x_1 \in [x]$ and ending in $[y]$ is independent of the choice of $x_1 \in [x]$. (It follows that the number starting in $[y]$ and ending at x_1 is also fixed.)

An equitable partition σ of X is necessarily a refinement of Σ. That is, if x has type i and y has type $j \neq i$, then x and y must be in different cells of σ because the number of i-arcs from x to $[x]$ is 1 and the number from y to $[x]$ is 0. In fact, σ induces an equitable partition on each cell of Σ. We say a cell of σ has type i if the vertices within that cell have type i.

Suppose σ is an equitable partition of X and let $r = |\sigma|$. Let $\overline{A}_i$ denote the r by r matrix with rows and columns indexed by the cells of σ, and $([x],[y])$ entry equal to the number of i-arcs from $x_1 \in [x]$ to $[y]$.

The product $\overline{A}_i \cdot \overline{A}_j$ has $([x],[y])$ entry equal to the number of i-j paths from $x_1 \in [x]$ to $[y]$. But this is just p_{ij}^k, counted for each k-arc joining x_1 to $[y]$ and for each k. Thus

$$\overline{A}_i \cdot \overline{A}_j = \sum_{k=0}^{d} p_{ij}^k \overline{A}_k \tag{2.2}$$

and $\mathbb{A}/\sigma := \langle \overline{A}_i \rangle_{0\leq i\leq d}$ is closed under multiplication. Note that commutativity (symmetry) of $\mathbb{A}$ implies that of $\mathbb{A}/\sigma$.

The main result of [GM] answers the question of when the algebra $\mathbb{A}/\sigma$ is itself the Bose-Mesner algebra of an association scheme.

3. Multisets

Another way to characterize this situation is to define the multiset joining $[x]$ to $[y]$. Let

$$\lambda := \{0^{\mu_0^\lambda}, 1^{\mu_1^\lambda}, \ldots, d^{\mu_d^\lambda}\}$$

where μ_i^λ is the number of i-arcs joining $x_1 \in [x]$ to $[y]$. The assumption that σ is an equitable partition means that these multisets are well-defined. Observe that the μ_i^λ are precisely the distinct entries of $\overline{A}_i$.

Now for each multiset λ, let M_λ be the matrix with rows and columns indexed by the cells of σ and $([x],[y])$ entry equal to 1 if λ is the multiset joining $[x]$ to $[y]$, 0 otherwise. Let Λ be the set of distinct multisets. M_λ is related to the $\overline{A}_i$ by:

$$\overline{A}_i = \sum_{\lambda\in\Lambda} \mu_i^\lambda M_\lambda.$$

Put $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. It will turn out, in the case that $\mathbb{A}/\sigma$ is the coherent algebra of a cc, that Λ is the set of basic relations.

Suppose λ is the multiset joining a cell $[x]$ to itself. Then λ contains exactly one element of Ω, hence λ does not join any two distinct cells. There is therefore a unique subset Λ_0 of Λ such that

$$\sum_{\lambda\in\Lambda_0} M_\lambda = I.$$

We have also, by definition,

$$\sum_{\lambda\in\Lambda} M_\lambda = J.$$

We have shown

Lemma 3.1. *$\mathcal{M}$ satisfies (i) and (ii) of 2.1.*

We shall see that 2.1 (iii) is satisfied if the cells of σ with the same type are *pairwise isometric* and (iv) is satisfied if the cells are *simple*. These two conditions are important in [GM] and are defined below in the terminology of multisets.

Given $\lambda \in \Lambda$, M_λ induces a digraph on the cells of σ, defined as follows. Make $[x]$ adjacent to $[y]$ iff λ joins $[x]$ to $[y]$. We say two cells $[x]$ and $[y]$ of σ are *isometric* if for all λ, the out-degree of $[x]$ in this induced digraph is the same as that of $[y]$. Observe that isometric cells must be cells of the same type, or *like cells*. Isometric cells, in other words, look the same locally.

Lemma 3.2. *If like cells of σ are pairwise isometric then $\mathcal{M}$ satisfies 2.1 (iii).*

Proof. We show first that isometric, like cells of σ have the same size. Let $[x]$ and $[y]$ be two such cells. Then $[x]$ is joined to itself by some multiset, say λ, and $[y]$ likewise. Now $|\lambda|$ is the number of points in $[x]$, and also the number of points in $[y]$.

Next, we claim Λ inherits a pairing from the pairing on the colors of $\mathcal{A}$. That is, if

$$(A_i)^T = A_{i^*} \quad (0 \le i \le d)$$

then we define λ^* as follows. Let h_α denote the size of a cell of σ of type α. Suppose λ joins a cell $[x]$ of type α to a cell $[y]$ of type β. Put

$$\mu_{i^*}^{\lambda^*} := \frac{h_\alpha}{h_\beta}\mu_i^\lambda.$$

We show λ^* is the multiset joining $[y]$ to $[x]$. The total number of i-arcs originating in $[x]$ and ending in $[y]$ is

$$\big|[x]\big| \cdot \mu_i^\lambda.$$

By reversing arcs, this equals the total number of i^*-arcs from $[y]$ to $[x]$. So

$$\big|[x]\big| \cdot \mu_i^\lambda = \big|[y]\big| \cdot (\#\ i^*\text{-arcs joining } y_1 \in [y] \text{ to } [x])$$

which shows λ^* is the multiset joining $[y]$ to $[x]$. Therefore,

$$M_\lambda^T = M_{\lambda^*}. \quad \square$$

The second condition requires the notion of an induced partition. For our purposes, this will always be induced from a cell of σ, but this is not necessary in the definition. Let $[z]$ be a cell of σ, and write $e_{[z]}$ for the characteristic vector of $[z]$ with respect to the underlying set of cells of σ. This is an r by 1 column vector with a single nonzero entry. (Note this differs from [GM].) Now form

$$\mathcal{D}([z]) := \big\langle \overline{A}_i \cdot e_{[z]} \big\rangle_{0 \le i \le d}$$

and define $\pi = \pi_{[z]}$, the partition of the cells of σ induced by $[z]$, as follows. Put $[x]$ and $[y]$ in the same cell of π if and only if every element in $\mathcal{D}([z])$ agrees in these two entries.

We call $[z]$ a *simple* cell if the dimension of $\mathcal{D}([z])$ is equal to the number of cells in $\pi_{[z]}$. It is pointed out in [GM] that $[z]$ is simple $\Leftrightarrow$ $\mathcal{D}([z])$ is closed under multiplication and contains the constants $\Leftrightarrow$ $\mathcal{D}([z])$ equals the span of the characteristic vectors for the cells of π. Since $\mathcal{D}([z])$ is always contained in this span, the third statement above really means that $\mathcal{D}([z])$ *contains* the characteristic vectors of cells of π.

The colors of a cc are confined to one block of the color matrix. That is, each color relates vertices of one fixed type to vertices of another fixed type. A consequence of this is that the multisets are disjoint unless they join pairs of cells of the same types. Let Λ_i be the subset of Λ containing all multisets joining *to* cells of type i. Then $\Lambda = \dot{\bigcup\limits_{i \in \Omega}} \Lambda_i$ defines a partition of Λ.

Let $[z]$ be a cell of σ of type i.

Lemma 3.3. *There is a one-to-one correspondence between the set of cells of* $\pi_{[z]}$ *and the set* Λ_i.

Proof. We claim cells $[x]$ and $[y]$ are joined to $[z]$ by the same multiset iff they are in the same cell of $\pi_{[z]}$. Entry $[x]$ of $\overline{A}_i \cdot e_{[z]}$ is equal to the number of i-arcs joining $x_1 \in [x]$ to $[z]$. If $[x]$ and $[y]$ have identical entries for all i, then they are joined by the same multiset to $[z]$. Conversely, if joined by the same multiset, the entries in $\overline{A}_i \cdot e_{[z]}$ match, so $[x]$ and $[y]$ are in the same cell of π. $\square$

Observe that the characteristic vector for a cell of $\pi_{[z]}$ is the $[z]$-column of M_λ for some $\lambda \in \Lambda$. (The multiset determined by the bijection above.) This shows:

Cor. 3.4. $[z]$ *is simple iff* $\mathcal{D}([z])$ *contains column* $[z]$ *of* M_λ*, for all* λ.

4. Quotient coherent configurations

In the theorem below, M_λ, $\overline{A}_i$ and $\mathbb{A}/\sigma$ are defined as in previous sections. Fix an ordering of Λ, let $r = |\Lambda|$ and define the $d+1$ by $r+1$ matrix M by

$$M_{ij} := \mu_i^{\lambda_j} \quad (0 \le i \le d,\ 0 \le j \le r).$$

(Row i of M gives the coefficients of $\overline{A}_i$ written as a linear combination of the multiset matrices.) The rows of M^T are the same as the distinct rows of the outer distribution matrix of $[z]$ ([GM], [BCN]).

Theorem 4.1. *Let $\mathcal{A}$ be a coherent configuration with coherent algebra $\mathbb{A}$, σ an equitable partition of $\mathcal{A}$ with like cells pairwise isometric. Then (i)–(v) below are equivalent.*

(i) *M has a left inverse.*
(ii) *The cells of σ are simple.*
(iii) *$\mathbb{A}/\sigma$ is a coherent algebra.*
(iv) *$M_\lambda \in \mathbb{A}/\sigma$ for all $\lambda \in \Lambda$.*
(v) *$\{M_\lambda\}_{\lambda\in\Lambda}$ is the set of basic relations for a cc whose coherent algebra is $\mathbb{A}/\sigma$.*

Proof.

(i) $\Rightarrow$ *(ii)***.** Let C be a left inverse of M, say

$$C = \left(c_k^\lambda\right)_{\lambda,k}.$$

Then for all $\lambda, \nu \in \Lambda$,

$$\sum_{k=0}^{d} c_k^\lambda \mu_k^\nu = \delta_{\lambda\nu}.$$

Let $[z]$ be a cell of σ. With $\pi = \pi_{[z]}$ as defined above, we show that the $[z]$-column of M_λ is in $\mathcal{D}([z])$, then apply Cor. 3.4. We claim

$$M_\lambda e_{[z]} = \sum_k c_k^\lambda \overline{A}_k e_{[z]}.$$

The $[x]$ entry of the sum on the right is

$$\begin{aligned} &\sum_k c_k^\lambda \cdot (\#\ k \text{ from } x_1 \in [x] \text{ to } [z]) \\ =&\sum_k c_k^\lambda \cdot \mu_k^\nu \qquad \text{where } \nu \text{ is the multiset joining } [x] \text{ to } [z] \\ =&\delta_{\lambda\nu}. \end{aligned}$$

Thus the sum on the right is the characteristic vector $M_\lambda e_{[z]}$.

(ii) $\Rightarrow$ *(i)***.** Assuming $[z]$ is simple implies there are constants c_k^λ with

$$M_\lambda e_{[z]} = \sum_k c_k^\lambda \overline{A}_k e_{[z]}.$$

But then $\sum_k c_k^\lambda \mu_k^\nu = \delta_{\lambda\nu}$, and the matrix $C := \left(c_k^\lambda\right)_{\lambda,k}$ is a left inverse of M.

$(i) \Rightarrow (iv)$**.** Assume there is a matrix $C = \big(c_k^\lambda\big)_{\lambda,k}$ with $CM = I$. We claim

$$M_\lambda = \sum_k c_k^\lambda \overline{A}_k.$$

The $([x],[y])$ entry on the right is

$$\sum_k c_k^\lambda \mu_k^\nu$$

where ν is the multiset joining $[x]$ to $[y]$. Since this equals $\delta_{\lambda\nu}$, the sum on the right is M_λ.

$(iv) \Rightarrow (i)$**.** Suppose

$$M_\lambda = \sum_k c_k^\lambda \overline{A}_k.$$

For each $\nu \in \Lambda$, choose an ordered pair $([x],[y])$ joined by ν. The $([x],[y])$ entry on the right is $\sum_k c_k^\lambda \mu_k^\nu$, and this must equal $\delta_{\lambda\nu}$.

$(iv) \Leftrightarrow (v)$**.** Since $\mathbb{A}/\sigma \subseteq \langle M_\lambda \rangle_{\lambda\in\Lambda}$ always, (iv) implies

$$\mathbb{A}/\sigma = \langle M_\lambda \rangle.$$

Then by 2.2, $\mathcal{M}$ satisfies 2.1 (iv). We conclude that $\mathcal{M}$ is the set of basic relations of a cc, and its coherent algebra is $\mathbb{A}/\sigma$. $(v) \Rightarrow (iv)$ is immediate.

$(iii) \Leftrightarrow (v)$**.** Assuming (iii), $\mathbb{A}/\sigma$ has a basis, say

$$D := \{D_0, D_1, \dots, D_l\},$$

consisting of all $(0,1)$ matrices which are primitive with respect to the Schur product. That is, there are constants α_{ij} such that

$$\overline{A}_i \circ D_j = \alpha_{ij} D_j$$

(see [BCN], 2.6; [DGH1]). Clearly M_λ is primitive, but may not be in $\mathbb{A}/\sigma$. Each entry of $\overline{A}_i$ is μ_i^λ for some λ. Let λ, ν be distinct elements of Λ. If the support of D_j overlaps the support of both M_λ and M_ν, then we have a contradiction: $\mu_i^\lambda = \mu_i^\nu$ for all i, and then $\lambda = \nu$. Thus the support of D_j is contained in the support of M_λ for some λ. On the other hand, we claim for each nonzero entry of M_λ the corresponding entry must be nonzero in D_j, for some j. This follows because for a nonzero entry in M_λ, the corresponding entry is nonzero also in $\overline{A}_i$ for some i, and D is a basis for $\mathbb{A}/\sigma$. We have $M_\lambda = \sum D_j$, where the sum is taken over j in some subset of $\{0, \dots, l\}$. But then $M_\lambda \in \mathbb{A}/\sigma$, and since

$$\overline{A}_i \circ M_\lambda = \mu_i^\lambda M_\lambda,$$

we find $\mathcal{M}_\lambda \in D$. Finally, $\mathcal{M}$ spans $\mathbb{A}/\sigma$, so $\mathcal{M} = D$.

Now, the fact that $\mathcal{M}$ is a basis for $\mathbb{A}/\sigma$ means that for some constants $\overline{p}_{\lambda\nu}^\tau$,

$$M_\lambda \cdot M_\nu = \sum_{\tau\in\Lambda} \overline{p}_{\lambda\nu}^\tau M_\tau$$

hence (iv) of 2.1 is satisfied by $\mathcal{M}$. By 3.1 and 3.2, (i)—(iii) are also satisfied. We have shown these are the basic relations for a cc, and its coherent algebra is $\mathbb{A}/\sigma$. $(v) \Rightarrow (iii)$ is immediate. $\square$

Remark. $M_\lambda = \sum_i c_i^\lambda \overline{A}_i$ does not define the constants c_i^λ uniquely, unless $r = d$.

5. Example

This family of examples is based on [DGH3, Example 4.1]. Let $\mathcal{A}$ be a cc derived as follows from two symmetric (v, k, λ)-designs, say $\mathcal{D}_1$, $\mathcal{D}_2$, with the same point set P. The vertex set is the disjoint union of $P \times P$ (type 1 vertices) and $\{B_1 \times B_2 \mid B_i \text{ is a block of } D_i, (i = 1, 2)\}$ (type 2).

Two vertices of type 1 are equal, adjacent or non-adjacent, where adjacency is defined as having one coordinate in common. Number these colors of the cc 0, 1 and 2 respectively. The fiber of $\mathcal{A}$ with type 1 vertices is a symmetric, rank 3 cc, equivalent to a strongly regular graph $L_2(v^2)$.

The fiber with type 2 vertices is defined similarly, and we number the equality, adjacency, and non-adjacency colors 3, 4, and 5 respectively.

A type 1 vertex (P_1, P_2) is incident with, adherent to, or separated from a vertex $B_1 \times B_2$ of type 2. Number these relations 6, 7, 8 respectively. Incidence is defined in the obvious way. Adherent means exactly one of P_1 and P_2 is contained in the corresponding block. Separated means that P_i is not contained in B_i $(i = 1, 2)$. Let $9 = 6^*$, $10 = 7^*$, $11 = 8^*$.

$\mathcal{A}$ is an example of a strongly regular design of the second kind ([DGH3]). We now define an equitable partition σ of the vertex set of $\mathcal{A}$. Partition both types of vertices by first coordinate. The cells of σ are then cliques in the lattice graphs.

The multisets are:

1. Between two cells of type 1: $\lambda_0 := \{0^1, 1^{v-1}\}$ $\lambda_1 := \{1^1, 2^{v-1}\}$
2. Between two cells of type 2: $\lambda_2 := \{3^1, 4^{v-1}\}$ $\lambda_3 := \{4^1, 5^{v-1}\}$
3. Between type 1 and type 2: $\lambda_4 := \{6^k, 7^{v-k}\}$ $\lambda_5 := \{7^k, 8^{v-k}\}$
4. $\lambda_6 := \lambda_4^*$, $\lambda_7 := \lambda_5^*$.

It can easily be seen that $\mathcal{A}/\sigma$ is a cc of type $\begin{bmatrix} 2 & 2 \\ & 2 \end{bmatrix}$. It is essentially equivalent to the original pair of symmetric designs.

6. Parameters

Let $\mathcal{A}/\sigma$ be a quotient cc with relations given by multisets and notation as in 4.1. In particular, fix a set of constants c_k^λ. If k joins vertices of type α to vertices of type β, put $v_k := p_{kk^*}^\alpha$. Similarly, for each multiset λ, define $v_\lambda := \overline{p}_{\lambda\lambda^*}^\alpha$. To determine v_λ, we count v_k and apply 4.1 (i). That is,

$$v_k = \sum_\lambda v_\lambda \mu_k^\lambda.$$

Writing $v = [v_0, v_1, \dots, v_d]^T$ and $\overline{v} = [v_{\lambda_0}, v_{\lambda_1}, \dots, v_{\lambda_r}]^T$ we have $v = M\overline{v}$. Multiplying both sides by C implies

$$v_\lambda = \sum_k c_k^\lambda v_k. \tag{6.1}$$

The parameters $\overline{p}_{\lambda\nu}^\tau$ are defined by the products

$$M_\lambda M_\nu = \sum_\tau \overline{p}_{\lambda\nu}^\tau M_\tau.$$

Then $\overline{p}_{\lambda\nu}^\tau$ is of course the number of cells $[z]$ with $[x]$ joined by λ to $[z]$ and $[z]$ by ν to $[y]$, where $([x], [y])$ is any ordered pair joined by τ. These are related to the given cc parameters and the multiset constants as follows.

Lemma 6.2.

$$\overline{p}^{\tau}_{\alpha\beta} = \sum_{i,j,k} c_i^{\alpha} c_j^{\beta} \mu_k^{\tau} p_{ij}^{k}$$

Proof. Given $[x]$ joined to $[y]$ by τ, count all i-j paths from $x_1 \in [x]$ to $[y]$. First, p_{ij}^k counts i-j paths from x_1 to some $y_1 \in [y]$. Including all possibilities for $y_1 \in [y]$, we get

$$\sum_k \mu_k^{\tau} p_{ij}^{k}.$$

On the other hand, we may count i-j paths through $[z]$ for all possible $[z]$. This gives

$$\sum_{\lambda,\nu} \overline{p}^{\tau}_{\lambda\nu} \mu_i^{\lambda} \mu_j^{\nu}.$$

Equating these, we then make use of C to solve for $\overline{p}^{\tau}_{\lambda\nu}$.

$$\sum_k \mu_k^{\tau} p_{ij}^{k} = \sum_{\lambda,\nu} \overline{p}^{\tau}_{\lambda\nu} \mu_i^{\lambda} \mu_j^{\nu}$$

Multiply both sides by c_i^{α} and sum over i.

$$\begin{aligned}
\sum_i c_i^{\alpha} \left(\sum_k \mu_k^{\tau} p_{ij}^{k} \right) &= \sum_{\lambda,\nu} \sum_i c_i^{\alpha} \overline{p}^{\tau}_{\lambda\nu} \mu_i^{\lambda} \mu_j^{\nu} \\
\sum_{i,k} c_i^{\alpha} \mu_k^{\tau} p_{ij}^{k} &= \sum_{\lambda,\nu} \overline{p}^{\tau}_{\lambda\nu} \left(\sum_i c_i^{\alpha} \mu_i^{\lambda} \right) \mu_j^{\nu} \\
\sum_{i,k} c_i^{\alpha} \mu_k^{\tau} p_{ij}^{k} &= \sum_{\lambda,\nu} \overline{p}^{\tau}_{\lambda\nu} \delta_{\alpha\lambda} \mu_j^{\nu}
\end{aligned}$$

Now multiply by c_j^{β} and sum over j.

$$\begin{aligned}
\sum_j c_j^{\beta} \left(\sum_{i,k} c_i^{\alpha} \mu_k^{\tau} p_{ij}^{k} \right) &= \sum_{\nu} \overline{p}^{\tau}_{\alpha\nu} \sum_j c_j^{\beta} \mu_j^{\nu} \\
\sum_{i,j,k} c_i^{\alpha} c_j^{\beta} \mu_k^{\tau} p_{ij}^{k} &= \sum_{\nu} \overline{p}^{\tau}_{\alpha\nu} \delta_{\beta\nu} \\
\sum_{i,j,k} c_i^{\alpha} c_j^{\beta} \mu_k^{\tau} p_{ij}^{k} &= \overline{p}^{\tau}_{\alpha\beta} \quad \square
\end{aligned}$$

The fact that $\mathbb{A}/\sigma$ is Schur-closed implies

$$\mu_i^{\lambda} \mu_j^{\lambda} = \sum_k c_{ij}^{k} \mu_k^{\lambda}$$

where the c_{ij}^k are constants independent of λ. Indeed,

$$\begin{aligned}\overline{A}_i \circ \overline{A}_j &= \sum_{\lambda} \mu_i^\lambda \mu_j^\lambda M_\lambda \\ &= \sum_{\lambda,k} \mu_i^\lambda \mu_j^\lambda c_k^\lambda \overline{A}_k \\ &= \sum_{\tau,\lambda,k} \mu_i^\lambda \mu_j^\lambda c_k^\lambda \mu_k^\tau M_\tau\end{aligned}$$

We have shown

$$\mu_i^\lambda \mu_j^\lambda = \sum_k c_{ij}^k \mu_k^\lambda \qquad \text{where} \qquad c_{ij}^k = \sum_\nu \mu_i^\nu \mu_j^\nu c_k^\nu. \tag{6.3}$$

7. Remarks

(1) The standard partition Σ is itself an equitable partition, since the number of i-arcs from a vertex of type α to a cell of type β is $p_{ii^*}^\alpha \cdot p_{i\beta}^i$. The quotient modulo Σ is a trivial cc with 1-point fibers.

(2) The number of fibers in a quotient is the same as in the original cc. In particular, a cc affords a quotient which is an association scheme only if it is homogeneous (possibly non-commutative). The quotient is commutative iff for all mulitsets τ

$$\sum_k \mu_k^\tau p_{ij}^k = \sum_k \mu_k^\tau p_{ji}^k \quad (0 \le i, j \le d).$$

This occurs iff the number of i-j paths from $x_1 \in [x]$ to $[y]$ equals the number of j-i paths.

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