REGULAR WEIGHTS OF FULL RANK ON STRONGLY REGULAR GRAPHS

A. D. Sankey

Abstract. This work addresses the existence question for weighted coherent configurations in the special case of rank 3, symmetric cc's, which are equivalent to strongly regular graphs, and weights of values ± 1. Examples related to rank 3 group actions of A_5, $S_4(q)$ and $PSL_3(4)$ are discussed and a complete account of the regular full rank weights on the lattice graph $L_2(n)$ is given. These are found to be either trivial or tensor products of pairs of weights whose coboundaries are regular 2–graphs.

0. Introduction

The theory of regular weights on coherent configurations (cc's) was introduced by D. G. Higman and developed in [8]. Cc's are combinatorial objects which are rife with algebraic structure. The adjacency algebra of a cc, or *coherent algebra* is parametrized by its invariants. Analysis of feasible parameter sets provides a starting point for investigation of the existence question. Much of the algebraic structure carries over to the weighted configurations, so the weighted adjacency algebra can be similarly parametrized.

There is a class of regular weights on rank 2 cc's which are equivalent to regular 2-graphs. These have been studied by D. E. Taylor ([15]). The generalization to regular weights of higher rank is thus a natural extension of that work. J. J. Seidel, D. E. Taylor and others have studied the connection between 2-graphs and sets of equiangular lines ([13,14]). Associated with a weighted adjacency algebra is a system of lines meeting with 2 intersection angles. Bounds are known for the

---

Research supported by the Department of Education and the Rackham Graduate School of the University of Michigan as part of Ph.D. thesis written under the direction of D. G. Higman.

Travel supported by Slippery Rock University and Jerusalem Combinatorics '93.

The author is greatly indebted to the referee for several careful readings and many good suggestions for improvement.

number of such lines in real and complex n-space ([5]). Some examples discussed here achieve these bounds.

In this work, we investigate the special case of symmetric cc's of rank 3, which are equivalent to strongly regular graphs (srg's), and weights with values ± 1. The only exceptions to this are certain group-theoretic examples which have non-real values in U_4, the group of fourth roots of unity. The necessary definitions for this special case are given here. For the general theory the reader is referred to [8]. There are substantial differences, due to a sincere attempt to make the present paper self-contained and no more complicated than necessary.

Investigation of feasible regular weight parameter sets for this case shows that they are relatively rare in occurrence ([10]). This suggests that searching for classification theorems for regular weights on the known families and types of srg's is likely to be worthwhile. We begin this process in the present paper with two approaches. The first is to find examples which arise in connection with rank 3 group actions. The second is combinatorial in nature and results in the classification of regular weights on one infinite family of srg's.

In section 1, we give definitions and background information on srg's, weights, monomial representations, and 2-graphs. The second section introduces our first concrete example, a regular weight on the triangular graph $T(5)$. This has been explicitly considered before and is mainly included for completeness. Section 3 concerns examples related to rank 3 group actions. It includes a list of rank 3 candidates, a discussion of an infinite family of regular weights connected with the symplectic groups $PSp_4(q)$ for q odd; and a description of a regular weight associated with the group $2.L_3(4)$. Regular weights on lattice graphs are addressed in section 4, which contains our main result. We show a non-trivial one occurs only for even n, and is the tensor product of two weights whose coboundaries are regular 2–graphs with identical parameters.

1. Preliminaries

1.1. Strongly regular graphs.

Definition. A *strongly regular graph* (srg) is a regular graph which is neither null nor complete, with the property that the number of vertices adjacent to two distinct vertices x and y depends only on whether or not x and y are adjacent.

The *parameters* of an srg Γ are (n, k, λ, μ), where n is the number of vertices, k is the valency, λ is the number of common neighbors to two adjacent vertices and μ is the number of common neighbors to two non-adjacent vertices.

The *complement* of an srg Γ is denoted by $\overline{\Gamma}$, and is also strongly regular, with parameters $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$. For convenience, we write $l = n-k-1$ for the valency of $\overline{\Gamma}$.

Examples.

(1) The pentagon is strongly regular with parameters $(5, 2, 0, 1)$.
(2) The triangular graph $T(n)$ $(n \geq 4)$ has unordered pairs from an n-set as vertices, with two adjacent iff they have an element in common. The srg parameters for $T(n)$ are $\big(\binom{n}{2}, 2(n-2), n-2, 4\big)$. The complement of $T(5)$ is the well-known Petersen graph.
(3) The lattice graph $L_2(n)$ $(n \geq 3)$ is pictured as an $n \times n$ grid. The vertices are the n^2 lattice points, with two adjacent iff they have one coordinate in common. The srg parameters for $L_2(n)$ are $\big(n^2, 2(n-1), n-2, 2\big)$.

Notation. Let Γ be an srg with identity, adjacency and non-adjacency relations given by f_0, f_1, f_2 respectively. Letting X be the vertex set, we say $(x, y, z) \in X^3$ is a triangle of *type* (i, j, k) if $(x, y) \in f_i$, $(y, z) \in f_j$ and $(x, z) \in f_k$. Denote by $A_0 = I_n, A_1$ and A_2 the adjacency matrices of the three relations. The *adjacency algebra* $\mathcal{A}$ is the linear span over $\mathbb{C}$ of $A_i, i \in I$, where I is the index set $\{0, 1, 2\}$. The *intersection numbers* for the adjacency algebra are the structure constants p_{ij}^k

defined by

$$A_i \cdot A_j = \sum_{i=0}^{2} p_{ij}^k A_k.$$

Observe that the parameters k, λ, μ are the structure constants $p_{11}^0, p_{11}^1, p_{11}^2$ respectively. The *intersection matrices* are M_j $(j \in I)$ defined by

$$M_j := \left(p_{ij}^k\right)_{i,k \in I}.$$

The regular representation $A_j \mapsto M_j$ is an isomorphism of $\mathcal{A}$ onto a (commutative) subalgebra of $M_3(\mathbb{C})$. This provides a useful tool for analyzing the adjacency algebra. Specifically, we make use of the fact that $\mathcal{A}$ is semi-simple, so it can be decomposed as a direct sum of simple ideals. The constituents of the standard character are linear since $\mathcal{A}$ is commutative. These are found using the 3 by 3 intersection matrices, which have the same eigenvalues as the adjacency matrices. Knowing that the multiplicities of the eigenvalues of A_i must be positive integers leads to an additional condition on potential parameter sets for srgs. (See [1], [3] for details.)

1.2. Regular weights. ([8])

Let U_4 be the set of fourth roots of unity. With respect to a given vertex set X, we define a *weight with values in* U_4 as a function $\omega : X^2 \to U_4$ with the properties:

(1) $\omega(x,x) = 1 \quad \forall x \in X$,

(2) $\omega(x,y) = \overline{\omega(y,x)} \quad \forall x, y \in X$.

Equivalently, we may view ω as a matrix indexed by X which is Hermitian and has unit diagonal.

If ω is a weight, then $\delta\omega : X^3 \to U_4$ is defined by

$$\delta\omega(x,y,z) := \omega(x,y)\overline{\omega(x,z)}\omega(y,z).$$

Thus $\delta\omega$ assigns a value to each triple of points or *triangle* (x,y,z).

Let Γ be an srg with vertex set X and relations f_i, $(0 \le i \le 2)$. Given $x \in X$, define $f_i(x) := \{y \in X | (x,y) \in f_i\}$. We now define parameters which link the weight ω and the srg Γ. For $x, z \in X$, and $\alpha \in U_4$, set

$$\beta_{ij}^{\delta\omega}(x,z,\alpha) := \left| \ \left\{ y \in f_i(x) \cap f_j(z) \ \middle| \ \delta\omega(x,y,z) = \alpha \right\} \ \right|$$

Definition. If ω is a weight with values in U_4 then ω is *regular* on Γ if for $(x,z) \in f_k$, $\beta_{ij}^{\delta\omega}(x,z,\alpha)$ is independent of the choice of $(x,z) \in f_k$.

Note. As defined in ([8]), a weight may take on the value 0, provided it vanishes completely on some subset of the relations, and does not take on the value 0 otherwise. The *rank* of ω is then defined as the number of relations on which ω does not vanish. In our case, the definitions have been formulated so that the rank is always 3 and this is what is meant by "full rank". The methods discussed here do, however, give rise to some regular weights which vanish on the srg or on its complement, and hence have rank 2.

Parameters. If ω is regular on Γ, we write $\beta_{ij}^k(\alpha) = \beta_{ij}^{\delta\omega}(x,z,\alpha)$. Then, given $(x,z) \in f_k$, $\beta_{ij}^k(\alpha)$ is the number of triples (x,y,z) of type (i,j,k) and weight α. Observe that $\beta_{ij}^k(\alpha)$ is nonnegative, integral, and bounded above by p_{ij}^k. Also, $\beta_{ij}^k(\alpha)$ is invariant under switching. To switch ω on a vertex x is to multiply row and column x by -1.

Viewing ω as a matrix, form the entry-wise product

$$A_j^\omega := \omega \circ A_j$$

for each j. These are the *weighted adjacency matrices.* We define

$$\beta_{ij}^k = \sum_{\alpha \in U_4} \alpha \beta_{ij}^k(\alpha).$$

From [8], we have (i) the weighted adjacency matrices span a subalgebra of $M_n(\mathbb{C})$. This *weighted adjacency algebra* is denoted $\mathcal{A}^\omega$; (ii) the β_{ij}^k are the structure constants for $\mathcal{A}^\omega$. Since $\mathcal{A}^\omega$ is semi-simple, the discussion in [7] of feasible traces applies to the trace character ζ of $\mathcal{A}^\omega$. The regular representation

$$A_j^\omega \mapsto M_j^\omega := \left(\beta_{ij}^k\right)_{i,k \in I} \ (j \in I)$$

as before is an isomorphism of commutative matrix algebras, and we make use of this in analyzing the constituents of ζ.

We call a regular weight ω *trivial* if for all indices j, $A_j^\omega = \pm A_j$.

With the exception of section 3.3, the weights considered here will have values in $U_2 = \{\pm 1\}$. This means of course that ω is real-valued and Hermitian, hence symmetric when viewed as a matrix. The properties listed below follow from the definitions when restricting to this case. In particular, if ω is a weight with values in $\{\pm 1\}$ and regular on the srg Γ, then the weighted intersection matrices must have the form

$$M_0 = I \quad M_1^\omega = \begin{pmatrix} & 1 & \\ k & A & B \\ & C & D \end{pmatrix} \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & C & D \\ l & E & F \end{pmatrix}.$$

Blank spaces indicate zero entries.

Observations. (See [8].)

(1) $\beta_{ij}^k \in \mathbb{Z}$.

(2) $\beta_{ij}^k(\alpha) = \beta_{ji}^k(\alpha)$.

(3) $\beta_{0j}^k = \beta_{0j}^k(1) = \delta_{jk}$.

(4) $\beta_{ij}^0 = \beta_{ij}^0(1) = \delta_{ij} v_i$, where v_i is the valency of the graph of f_i.

(5) $|\beta_{ij}^k| \le p_{ij}^k$.

(6) $C = \frac{Bl}{k}, E = \frac{Dl}{k}$.

To see this last remark, note that for a fixed vertex x, the number of triangles (x, y, z) of type $(2, 1, 1)$ and weight α is the same as the number of triangles (x, z, y) of type $(1, 1, 2)$ and weight α. Counting these gives $kC = lB$. Similarly, $kE = lD$.

A set of *feasible regular weight parameters* for a given srg is the set of integers $\{A, B, \dots, F\}$ which satisfy the constraints above and such that the multiplicities of the eigenvalues of M_j^ω $(j \in I)$ are positive integers. Since the regular weight parameters are bounded by the srg parameters, a list of feasible regular weight parameters for a given srg is easily generated by computer. Furthermore, the well-known conditions on feasible srg parameters make it straight-forward to generate all feasible srg parameters up to a chosen maximum number of vertices.

Weighted adjacency algebras. (See [7], [8] for more detail.) The weighted adjacency algebra $\mathcal{A}^\omega$ is semi-simple, so the earlier discussion of decomposition into simple ideals applies to this case as well. The eigenvalues of A_i^ω and their multiplicities are listed in a *character-multiplicity table*.

The standard character ζ decomposes into linear constituents:

$$\zeta = \sum_{i=0}^{2} z_i \zeta_i.$$

$\mathcal{A}^\omega$ has a basis of pairwise orthogonal idempotents given by

$$\epsilon_i = z_i \sum_j \frac{1}{nv_j} \zeta_i(A_j^\omega) A_j^\omega.$$

(This is true of the adjacency algebra also, but we make use of it only in the weighted case.) Observe that the matrix $\frac{n}{z_i}\epsilon_i$ is positive semi-definite, symmetric, with unit diagonal and is therefore the Gram matrix of a set of n vectors in z_i-dimensional Euclidean space. We have a system of n lines in z_i-space meeting with 2 intersection angles:

$$\cos\theta_1 = \pm\left|\frac{\zeta_i(A_1^\omega)}{k}\right| \qquad \cos\theta_2 = \pm\left|\frac{\zeta_i(A_2^\omega)}{l}\right|.$$

Delsarte, Goethals and Seidel give bounds for the number of lines in real and complex n-space meeting with 2 intersection angles ([5]). Some of our examples achieve these bounds.

1.3. Monomial representations. The so-called *rank 3* srg's, which are afforded by certain group actions, are discussed in section 3. Here, we give background information on monomial representations which is needed later. ([11] is one of many good sources for information on group representations.)

A *monomial matrix* is a square matrix with exactly one nonzero entry in each row and column. A *monomial representation* of a group G is a representation of G into a group of monomial matrices. Associated with a monomial representation is the *underlying permutation representation.* This is determined by replacing all nonzero entries by 1's in each matrix of the representation. If this underlying action of the group is transitive, we say the monomial representation is transitive.

Let Γ be a transitive monomial representation of a finite group G. Then Γ is equivalent to the induced representation of a degree 1 representation λ of a subgroup

H ([11]). (In our case, H will be the stabilizer of a point in the underlying group action.) Let $n = G : H$ and fix a transversal $\{t_1, t_2, \dots, t_n\}$ to H in G. Denote the induced representation by λ^G. For $g \in G$, $\lambda^G(g)$ is the matrix with ij entry

$$\lambda^0(t_i g t_j^{-1})$$

where $\lambda^0(h) = \lambda(h)$ if $h \in H$ and $\lambda^0(h) = 0$ otherwise.

1.4. 2-graphs.

Definition. A 2–*graph* is a set of vertices X and a distinguished set of coherent triples $\Delta \subseteq X^3$ with the property that every 4–subset of X contains an even number of triples in Δ.

Every graph gives rise to a 2-graph as follows ([13, 15]). Let X be the vertex set and think of the graph as a function $f : X \times X \to \{0, \pm 1\}$ where the value assigned to adjacent pairs is $+1$; non-adjacent pairs -1; and the diagonal is mapped to 0. Assign a value to each triple (x, y, z) by $\delta f : X^3 \to \{0, \pm 1\}$ with

$$\delta f(x, y, z) = f(x, y) f(x, z) f(y, z).$$

Then define

$$\Delta := \{(x, y, z) \mid \delta f(x, y, z) = -1\}.$$

This 2-graph is sometimes referred to as δf. The $(0, \pm 1)$ adjacency matrix for f is said to *represent* the 2-graph, though it is by no means unique in that regard. Any matrix obtained from it by switching represents the same 2-graph.

Definition. A 2–graph Φ is *regular* if and only if every pair of vertices is contained in the same number of coherent triples.

Srg's are by definition not complete, but we may define a regular weight on the complete graph similarly. These regular weights are equivalent to regular 2–graphs: take the set of coherent triples to be all those with weight -1.

The *parameters* of a non-trivial regular 2–graph are (n, a, b), where n is the number of vertices, a is the number of coherent triples containing a given pair and

b is the number of coherent 4-sets containing a given coherent triple, where a 4–set is coherent if and only if all of its 3–subsets are. It is shown in [15] that both n and a are even.

2. Example

The complement of the triangular graph $T(5)$— known as the Petersen graph— is realized in euclidean 3-space by the faces of a regular icosahedron. This example in its geometric context is discussed by J.J. Seidel in [12], where he calls it "the prototype for weighted coherent configurations". D.G. Higman describes the same example explicitly in the context of monomial representations ([8]).

Take the ten lines joining the centers of opposite faces as vertices, and call two vertices adjacent if and only if they meet at the smaller of two possible angles. Call this graph Γ. To define a regular weight on Γ, first choose a direction for each of the ten lines, $x_1, \dots, x_{10}$ and label the faces x_i^+, x_i^- according to this orientation. Define

$$\omega(x_i, x_j) = \begin{cases} +1 & \text{if like faces meet at acute angles,} \\ -1 & \text{if unlike faces meet at acute angles.} \end{cases}$$

(What is meant by this is that the angle between the line segments drawn from the faces to the center of the icosahedron is acute.) This is, up to switching, the only non-trivial regular weight on Γ ([8]).

A proof of regularity relying only on the geometry is given in [10]. We outline the proof as follows. Consider $(a, x_j) \in f_k$, and suppose we want to calculate $\delta\omega(a, x_i, x_j)$ for some x_i. Observe that $\omega(a^q, x_i^r) = qr$ and $\omega(a^q, x_j^s) = qs$ for some choice of r and s, that is whatever makes the acute angle with a^q. Then $\delta\omega(a, x_i, x_j)$ depends only on whether x_i^r and x_j^s meet at an acute angle— not on the actual signs r and s. It follows from the symmetry of the figure that $\beta_{lm}^{\delta\omega}(a, x_i, x_j)$ $(l, m \in I)$ is independent of the choice of $(a, x_j) \in f_k$.

Intersection matrices.

$$M_1^\omega = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \quad M_2^\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 6 & 0 & -1 \end{pmatrix}.$$

We get the (linear) characters and their multiplicities from M_1^ω and M_2^ω. The 10 by 10 weighted adjacency matrices are given in [8].

Character–multiplicity table for Γ.

	I	A_1^ω	A_2^ω	z_i
ζ_1	1	0	-3	4
ζ_2	1	$\sqrt{5}$	2	3
ζ_3	1	$-\sqrt{5}$	2	3

Remark. The line system associated with this example realizes the special bound given in [5], with 10 lines meeting in real 3-space.

3. The group case

Let G be a finite group which acts transitively on a set X with symmetric orbitals. Then the orbitals are the basic relations for an association scheme, which has rank equal to the number of orbitals. (So, this gives rise to an srg when the number of orbitals is 3.) Alternatively, the number of orbits for the stabilizer of a point is 3. The action of G on X is equivalent to the action of G on the cosets modulo a point stabilizer. When it is convenient, we will view the group action as the latter.

It is well known that the centralizer algebra of the matrices in the permutation representation is a coherent algebra. That is, the set of matrices in $M_n(\mathbb{C})$ which commute with the matrices of the permutation representation is the adjacency algebra of the cc formed by the orbitals. Loosely speaking, the permutation representation "corresponds" to the srg via the centralizer algebra. The following theorem, quoted from [8], states the analogous fact for the weighted adjacency algebra. It gives sufficient conditions for existence of regular weights on coherent configurations (cc's). We apply the theorem in the special case of rank 3, commutative cc's, and I_ω (the set of indices on which ω does not vanish) equal to $\{0, 1, 2\}$.

Theorem (D.G. Higman). *The centralizer algebra of a transitive monomial representation* Γ *of a finite group* G *is a weighted algebra* $\mathcal{A}^\omega$*, where* $\mathcal{A}$ *is the centralizer* *algebra of the underlying permutation representation, and ω is a regular weight on the cc afforded by the underlying action of G with $I_\omega = I_\Gamma$.*

In our case, we look for index 2 subgroups of a point stabilizer H to give rise to a regular weight with values ± 1. If A is such a subgroup, we let λ be the linear representation of H given by $\lambda(h) = 1$ if $h \in A$ and $\lambda(h) = -1$ otherwise. Then $\Gamma := \lambda^G$ is a monomial representation of G with underlying permutation representation which corresponds to an srg. By Higman's theorem, Γ gives rise to a regular weight ω on the srg. It may be that ω is trivial or has rank less than 3 or has values other than ± 1 in the group of fourth roots of unity.

3.1. Rank 3 candidates. A starting point in the search for concrete examples arising this way is the *Atlas of Finite Groups* ([4]) which contains character tables as well as information about rank 3 actions for many simple groups, including all sporadic groups and some members of each infinite family. The list of rank 3 candidates below includes all *Atlas* groups which have (i) a rank 3 action with a point stabilizer having an index 2 subgroup, and (ii) irreducible character degrees in G or in $2.G$ which match the eigenvalue multiplicities of a set of feasible regular weight parameters for the associated srg. Where the degrees are in parentheses, (ii) has not yet been determined. A bar over the character degree indicates non-real character values.

G	G_x	**index**	**degrees-**G	**degrees-**2.G
A_5	S_3	10	$3+3+4$	
$S_4(3)$	$3^3 : S_4$	40	$\overline{5}+\overline{5}+30$	
	$3_1^{1+2} : 2A_4$			
$L_3(4)$	A_6	56		$\overline{10}+\overline{10}+36$
$S_4(4)$	$L_2(16):2$	120	$18+51+51$	
A_{10}	$(A_5 \times A_5):4$	126	$9+42+75$	
$S_4(4)$	$(A_5 \times A_5):2$	136	$34+51+51$	
$S_4(5)$	$5_+^{1+2} : 4A_5$	156	$13+13+130$	
	$5^3 : (2 \times A_5){\cdot}2$			
$U_4(3)$	$3_+^{1+4}.2S_4$	280	$35+35+210$	
$G_2(3)$	$U_3(3):2$	351	$78+91+182$	
	$5^3 : (2 \times A_5){\cdot}2$			
$O_7(3)$	$2U_4(3):2_2$	351	$78+91+182$	
$G_2(4)$	$U_3(4):2$	2016	$(378+819+819)$	

Fi_{22}	$2{\cdot}U_6(2)$	3510		$(429+1001+2080)$
Fi_{23}	$O_8^+(3):S_3$	137632	$(5083+25806+106743)$	

The first entry on the list affords the $T(5)$ example discussed earlier. Examples are known for $A_5, S_4(3), L_3(4), A_{10}, S_4(5)$. $S_4(4)$ is work in progress, and the others, so far as the author is aware, are unknown.

3.2. Regular weights associated with $S_4(q)$.

Consider the group

$$Sp_4(q) = \{g \in Gl_4(q) \mid gEg^t = E\} \quad \text{where } E = \begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix}$$

for q an odd prime power. Let $G = S_4(q)$, that is the image of $Sp_4(q)$ under the natural map $Sp_4(q) \longrightarrow Sp_4(q)/\langle -I_4\rangle$. G acts rank 3 on the totally isotropic lines of the symplectic geometry, with parameters $(q^3+q^2+q+1, q(q+1), q-1, q+1)$, where two lines are adjacent iff they meet. We claim that the line stabilizer G_L has an index 2 subgroup, and that the associated monomial character has rank 3. It follows that there is a regular rank 3 weight on the srg.

Fix a symplectic basis $\{e_1, e_2, e_{-1}, e_{-2}\}$. Let L be the line $\langle e_1\rangle + \langle e_2\rangle$. Let $L_1 = \langle e_1\rangle + \langle e_{-2}\rangle$, $L_2 = \langle e_{-1}\rangle + \langle e_{-2}\rangle$, and take (L, L_1) and (L, L_2) as representatives for the 2 non-trivial orbitals under the action of G. For each $g \in Sp_4(q)$ we let $\overline{g}$ denote the image of g in G. The line stabilizer G_L is represented modulo $\{\pm I\}$ by matrices of the form

$$g = \begin{pmatrix} A & B \\ & A^{-t} \end{pmatrix}$$

with BA^t symmetric and A non-singular. Write

$$\begin{aligned} \begin{pmatrix} A & B \\ & A^{-t} \end{pmatrix} &= \begin{pmatrix} I & BA^t \\ & I \end{pmatrix} \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} \\ &= \begin{pmatrix} I & S \\ & I \end{pmatrix} \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} \end{aligned}$$

so it is clear that $G_L \simeq q^3 : L_2(q).\,(q-1)$ where $(q-1)$ is the cyclic group $\mathbb{F}_q^*$. Since q is odd, G_L has an index 2 subgroup H, and we may describe H in the following way. Given an element $g = \begin{pmatrix} A & B \\ & A^{-t} \end{pmatrix}$ of $Sp_4(q)$, its image $\overline{g}$ in G (which is obviously in G_L) is contained in $H \iff \det(A)$ is a square in $\mathbb{F}_q^*$.

We compute the rank 3 weight parameters β_{ij}^k.

Intersection matrices.

$$M_1^\omega = \begin{pmatrix} 0 & 1 & 0 \\ q(q+1) & 0 & \pm(q+1) \\ 0 & \pm q^2 & 0 \end{pmatrix}$$

$$M_2^\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \pm q^2 & 0 \\ q^3 & 0 & \pm q(q-1) \end{pmatrix}$$

"+" if $q \equiv 1 \pmod 4$ and "−" if $q \equiv 3 \pmod 4$.

Character–multiplicity table for $S_4(q)$.

	I	A_1^ω	A_2^ω	z_i
ζ_1	1	0	$\mp q$	q^3+q
ζ_2	1	$(q+1)\sqrt{q}$	$\pm q^2$	$\frac{1}{2}(q^2+1)$
ζ_3	1	$-(q+1)\sqrt{q}$	$\pm q^2$	$\frac{1}{2}(q^2+1)$

Line systems. Projection into either of the $(\frac{q^2+1}{2})$–dimensional eigenspaces gives us a system of q^3+q^2+q+1 lines in $(\frac{q^2+1}{2})$–space. The Gram matrix is given by

$$\frac{n}{z_i}\epsilon_i = \left[I + \frac{1}{\sqrt{q}} A_1^\omega \pm \frac{1}{q} A_2^\omega \right].$$

In two cases, we find that this line system realizes the special bound of Delsarte, Goethals and Seidel ([5]). For $q = 3$, we have 40 lines in complex 5-space meeting at squared cosines $\frac{1}{3}$, $\frac{1}{9}$. For $q = 5$, we have 156 lines in real 13-space meeting at squared cosines $\frac{1}{5}$, $\frac{1}{25}$. The special bound applies only for $q \leq 9$ in the real case, and for $q \leq 3$ in the complex case. The absolute bound, which is $\mathcal{O}(q^8)$, is in general much larger than the number of lines in our examples.

Generalizing to $S_{2n}(q)$. We generalize the examples above by considering the action of $G = S_{2n}(q)$ on the maximal totally isotropic subspaces. (For $2n = 4$, these are the totally isotropic lines.) Of course, the action has greater rank for greater n. If n is even, q odd, the stabilizer of a maximal totally isotropic subspace has an index 2 subgroup. The rank, parameters, etc. have yet to be investigated in this more general setting.

3.3. Regular weights associated with $2.L_3(4)$.

Let G be the projective special linear group $L_3(4)$. G permutes the ovals of $PG_2(4)$ in 3 orbits of length 56. The action on each orbit has rank 3, affording in each case the unique srg Γ with parameters $(56, 10, 0, 2)$ ([6]). This is known as the *Gewirtz graph*. The stabilizer of an oval is isomorphic to A_6. The three A_6's are non-conjugate in G, since each one intersects a different conjugacy class of elements of order 4 in G. Our notation in references to the characters and conjugacy classes of G will be consistent with that of [4]. Note that the action of G on its cosets modulo A_6 is equivalent to the action on ovals. Since A_6 has no index 2 subgroup, it is clear that there is no monomial representation of G with values ± 1 associated with this rank 3 action. Accordingly, we consider the double cover $\overline{G} = 2.G$. The inverse image of an A_6 splits in $\overline{G}$, as can be seen from the character table. Thus $\overline{G}$ contains a subgroup $H \simeq 2 \times A_6$. Let $A \leq H, A \simeq A_6$, and let λ be the linear character of H with kernel A. We claim that $\lambda^{\overline{G}}$ has rank 3 if A has non-empty intersection with the conjugacy class 4A. (It has rank 2 if A meets 4B or 4C.) It follows that there are regular weights of rank 2 and rank 3 on Γ.

Remark. The monomial representation of $\overline{G}$ induced from λ is faithful ($-I_6$ maps to $-I_{56}$). The underlying permutation representation is the action of $\overline{G}$ on cosets modulo H, and is transitive. This action is equivalent to the action of G on cosets modulo A, hence affords the Gewirtz graph.

To compute the rank of $\lambda^{\overline{G}}$, we work with the group theory language *Cayley* ([2]). Given generators for $\overline{G}$ as a matrix group, it is not difficult to find three non-conjugate copies of A_6 inside $\overline{G}$ using *Cayley*, and to compute the rank of $\lambda^{\overline{G}}$ and the character values explicitly in each case. Since $\overline{G} < 2.U_4(3)$, which is a quotient group of $K := 6.U_4(3)$, we begin with Lindsey's generators for K from [9], which are 6 by 6 matrices with entries in $\mathbb{Q}[\alpha]$, $\alpha = e^{2\pi i/3}$.

Let $\mathcal{O} = \mathbb{Z}[\alpha]$, the ring of integers in $\mathbb{Q}[\alpha]$. K is a group of linear transformations of an $\mathcal{O}$-module M. Since the ideal $(1-\alpha)$ contains the integer 3 and is irreducible in $\mathcal{O}$, we find that $\mathcal{O}/(1-\alpha) \simeq \mathbb{F}_3$, thus the action of K on $M/(1-\alpha)M$ is equivalent to the action on a module over $\mathbb{F}_3$. Replacing entries in the generators for K by appropriate elements of $\mathbb{F}_3$, we get 6 by 6 matrix generators for the group

$$K/\langle \alpha I\rangle \simeq 2.U_4(3).$$

To find a subgroup of this group which is isomorphic to $2.L_3(4)$, we enter the matrices into Cayley. The action of this group on the set of 1-dimensional subspaces $\langle(1,0,0,0,0,0)\rangle^{2.U_4(3)}$ provides a permutation group isomorphic to $U_4(3)$. (Computations are much faster in permutation groups.) A subgroup isomorphic to $L_3(4)$ is generated by elements from a Sylow 3-subgroup and a Sylow 7-subgroup. Finally, we pull back to the matrix group to get the generators for $\overline{G}$.

Generators of $2{\cdot}L_3(4)$.

$$S = \begin{pmatrix} 1 & & 1 & & -1 & -1 \\ & 1 & & 1 & 1 & 1 \\ & & & -1 & 1 & \\ & & 1 & -1 & -1 & -1 \\ & & & & 1 & -1 \\ & & & & & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} & & -1 & & 1 & \\ 1 & & -1 & -1 & & 1 \\ & & 1 & -1 & -1 & \\ & & & -1 & 1 & 1 \\ & & 1 & 1 & & -1 \\ & -1 & -1 & 1 & -1 & -1 \end{pmatrix}$$

Note the center of $\overline{G}$ is $\langle -I\rangle$. By the same method, a subgroup of G isomorphic to A_6, and intersecting the conjugacy class 4A, is found and pulled back to $\overline{G}$ to get $H \simeq 2 \times A_6$. This group is generated by

$$-I, \quad \begin{pmatrix} 1 & & & -1 & -1 & \\ & 1 & & -1 & & \\ & & 1 & & & -1 \\ & & 1 & -1 & -1 & 1 \\ & & 1 & 1 & & \\ & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 & 1 & & \\ -1 & -1 & -1 & -1 & & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & & 1 & -1 & 1 \\ -1 & -1 & & 1 & & 1 \end{pmatrix}.$$

We next define λ as described above, and calculate the character values of $\chi = \lambda^G$ by:

$$\chi_i = \frac{\overline{G} : H}{|C_i|} \sum_{x \in C_i \cap H} \lambda(x)$$

where C_i is the ith conjugacy class of $\overline{G}$ and χ_i is the value of χ on that class. From the character table for $\overline{G}$, we find $\chi = \chi_{11} + \chi_{12} + \chi_{15}$. Thus χ has 3 irreducible constituents with degrees 10, 10 and 36. Hence the regular weight ω associated with χ has rank 3. The degree 10 constituents are complex conjugates, which implies that ω is not real-valued. In fact, A_1^ω has entries $0, \pm i$ and A_2^ω has entries $0, \pm 1$.

Intersection matrices.

$$M_1^\omega = \begin{pmatrix} & 1 & \\ 10 & 0 & 2 \\ & 9 & 0 \end{pmatrix} \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & 9 & 0 \\ 45 & 0 & 4 \end{pmatrix}$$

Character–multiplicity table for $2.L_3(4)$.

	I	A_1^ω	A_2^ω	z_i
ζ_1	1	0	-5	36
ζ_2	1	$2\sqrt{7}$	9	10
ζ_3	1	$-2\sqrt{7}$	9	10

4. Regular weights on lattice graphs

The regular weights described in this section are not examples which arise in the group case. While the lattice graph $L_2(n)$ is afforded by a rank 3 action of $\Sigma_n \wr 2$, the regular weights obtained in this way are trivial.

4.1. A regular weight on $L_2(6)$.

Let Γ be the lattice graph $L_2(6)$. Let $A_1,\ A_2$ be adjacency matrices of Γ and $\overline{\Gamma}$ respectively. Put

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

C is called a conference matrix of order 6 ($C^2 = 5I$), and represents a regular 2–graph on 6 vertices ([13]). Number these according to the rows and columns of the lattice graph. Put

$$\begin{aligned} \omega &= (I + C) \otimes (I + C) \\ &= I \otimes I + I \otimes C + C \otimes I + C \otimes C. \end{aligned}$$

Now ω is a matrix of the same size as the adjacency matrices, so we may define $A_i^\omega = \omega \circ A_i$ for each $i \in I$.

Proposition. ω is regular on $L_2(6)$.

Proof. We have

$$\begin{aligned} A_0^\omega &= I \otimes I & (A_1^\omega)^2 &= 10A_0^\omega + 2A_2^\omega \\ A_1^\omega &= I \otimes C + C \otimes I & A_1^\omega A_2^\omega &= 5A_1^\omega \\ A_2^\omega &= C \otimes C & (A_2^\omega)^2 &= 25A_0^\omega. \end{aligned}$$

This gives us the weighted intersection matrices,

$$M_1^\omega = \begin{pmatrix} & 1 & \\ 10 & 0 & 2 \\ & 5 & 0 \end{pmatrix} \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & 5 & 0 \\ 25 & 0 & 0 \end{pmatrix}$$

which define $\{\beta_{ij}^k\}_{0 \le i,j,k \le 2}$. We get

$$\beta_{ij}^k(\pm 1) = \frac{1}{2}(p_{ij}^k \pm \beta_{ij}^k),$$

recalling that $\beta_{ij}^k(1) + \beta_{ij}^k(-1) = p_{ij}^k$ and $\beta_{ij}^k(1) - \beta_{ij}^k(-1) = \beta_{ij}^k$. We check easily that β_{ij}^k and p_{ij}^k have the same parity, so that $\beta_{ij}^k(\pm 1)$ is an integer. $\square$

Character–multiplicity table for $L_2(6)$.

	I	A_1^ω	A_2^ω	z_i
ζ_1	1	0	-5	18
ζ_2	1	$2\sqrt{5}$	5	9
ζ_3	1	$-2\sqrt{5}$	5	9

4.2. Regular weights on $L_2(n)$.

We quote a theorem from [15], used in the proof of the theorem below. A graph is *strong* if for any pair $x, y \in X$ the number of vertices adjacent to exactly one of x or y depends only on whether x and y are adjacent.

Theorem (D. E. Taylor). *Suppose f is a graph with $(0, \pm 1)$ adjacency matrix A. Then the 2–graph δf is regular if and only if f is a strong graph and the minimal polynomial of A is quadratic.*

The following theorem shows that all non-trivial regular rank 3 weights on $L_2(n)$ are obtained from regular 2–graphs as in the example above.

Theorem. *If ω is a non-trivial regular weight with full support on the lattice graph $L_2(n)$ then n is even and $\omega = \omega_1 \otimes \omega_2$, where $\delta\omega_1$ and $\delta\omega_2$ are regular 2–graphs with the same parameters.*

Proof. Let ω be a rank 3 weight regular on the lattice graph $\Gamma = L_2(n)$. Let $M_i^\omega = M_i^\omega(1) - M_i^\omega(-1)$ $(i = 1, 2)$ be the intersection matrices for the weighted adjacency algebra $\mathcal{A}^\omega$, and let C_1^ω, C_2^ω be the weighted $n^2 \times n^2$ adjacency matrices for Γ and $\overline{\Gamma}$ respectively. Number the vertices of Γ along rows and down columns. The adjacency matrices for Γ have the form

$$C_1 = I \otimes (J - I) + (J - I) \otimes I$$

$$C_2 = (J - I) \otimes (J - I).$$

where I and J are the $n \times n$ identity and all ones matrix respectively. Each of C_1 and C_2 contains n^2 blocks, where each block is an n by n matrix equal to either I or $J - I$. We may number the rows and columns of blocks from 1 to n and refer to the ij–block of the matrix.

We may assume that ω is switched so that non-zero entries in the first row and column are $+1$. The ordinary intersection matrices for Γ are

$$M_1 = \begin{pmatrix} & 1 & \\ 2(n-1) & n-2 & 2 \\ & n-1 & 2(n-2) \end{pmatrix}$$

$$M_2 = \begin{pmatrix} & & 1 \\ & n-1 & 2(n-2) \\ (n-1)^2 & (n-1)(n-2) & (n-2)^2 \end{pmatrix}.$$

Put

$$M_1^\omega(1) = \begin{pmatrix} & 1 & \\ 2(n-1) & a & b \\ & c & d \end{pmatrix}, \quad M_2^\omega(1) = \begin{pmatrix} & & 1 \\ & c & d \\ (n-1)^2 & e & f \end{pmatrix}$$

and

$$M_1^\omega = \begin{pmatrix} & 1 & \\ 2(n-1) & A & B \\ & C & D \end{pmatrix}, \quad M_2^\omega = \begin{pmatrix} & & 1 \\ & C & D \\ (n-1)^2 & E & F \end{pmatrix}$$

where

$$\begin{array}{ll} A = 2a-(n-2), & D = 2d-2(n-2), \\ B = 2b-2, & E = \frac{D(n-1)}{2}, \\ C = \frac{B(n-1)}{2}, & F = 2f-(n-2)^2, \end{array}$$

and a, b, d, f lie between 0 and the corresponding entries in M_1 and M_2. Now $c \in \mathbb{Z} \Rightarrow b$ is even or n is odd. But $0 \le b \le 2$, and we may take $1 \le b \le 2$, replacing A_i^ω by $-A_i^\omega$ $(i = 1, 2)$ if necessary. We first show that assuming $b = 1$ leads to a contradiction.

Suppose $b = 1$. Then $c = \frac{n-1}{2}$, and n is odd. We claim that every rectangle in Γ has 3 sides of equal weight. (Or, every rectangle has either one or three sides of weight $+1$.) To see this, choose non-adjacent vertices x and z. There are 2 triangles (x, y, z) of type $(1, 1, 2)$, and they form a rectangle (x, y_1, z, y_2) in Γ. Since $b = 1$, one of these must have weight 1 and the other weight -1. So the rectangle with vertices x, y_1, z, y_2 has the property

$$\omega(x, y_1) = \omega(y_1, z) \iff \omega(x, y_2) \neq \omega(y_2, z),$$

proving the claim. We will call this rectangle property "RP".

Given adjacent vertices x and z, $c = \frac{n-1}{2}$ implies that half of the triangles (x, y, z) of type $(2, 1, 1)$ have weight 1 and half have weight -1. Reordering the columns if necessary, we may assume that

$$\omega(n+1, n+i) = \begin{cases} 1 & \text{if } 2 \le i \le \frac{n+1}{2} \\ -1 & \text{if } \frac{n+1}{2} < i \le n. \end{cases}$$

Consider rectangles with vertices $n+1, 1, i, n+i$. By RP, we have

$$\omega(i, n+i) = \begin{cases} -1 & \text{if } 2 \le i \le \frac{n+1}{2} \\ 1 & \text{if } \frac{n+1}{2} < i \le n. \end{cases}$$

Note in particular that $\omega(2, n+2) = -1$. Set $a_i = \omega(n+2, n+i)$, $(3 \le i \le n)$. Applying RP to rectangles with vertices $2, n+2, i, n+i$ we have

$$\omega(2, i) = \begin{cases} -a_i & \text{if } 3 \le i \le \frac{n+1}{2} \\ a_i & \text{if } \frac{n+1}{2} < i \le n. \end{cases}$$

Now consider triangles $(1, i, 2)$ and $(n+1, n+i, n+2)$ of type $(1,1,1)$. Clearly

$$\delta\omega(1, i, 2) = -\delta\omega(n+1, n+i, n+2),\ (3 \le i \le n).$$

But then

$$\begin{aligned} a &= \left|\{i \mid \delta\omega(1,i,2) = 1\}\right| \\ &= \left|\{i \mid \delta\omega(n+1, n+i, n+2) = -1\}\right| \\ &= n - 2 - a. \end{aligned}$$

So $2a = n-2$, contradicting the fact that n is odd. Thus $b = 1$ is impossible.

We have $b = 2$, which forces $B = 2$ and $C = n-1$. Thus all triangles of types $(1,1,2)$ and $(2,1,1)$ have weight 1. We claim that all rectangles in Γ have an even number of sides with weight $+1$. Given x and z non-adjacent, consider the 2 triangles (x, y, z) of type $(1,1,2)$. Since both have weight $+1$, $\omega(x, y_1) = \omega(y_1, z) \iff \omega(x, y_2) = \omega(y_2, z)$. Call this new rectangle property "RP".

Let C_{ij}, $\overline{C}_{ij}$ be the ij–blocks of C_1^ω, C_2^ω respectively. We will show there are matrices P and Q such that

$$\begin{aligned} C_1^\omega &= I \otimes P + Q \otimes I, \\ C_2^\omega &= Q \otimes P. \end{aligned}$$

Step 1. Show $C_{1i} = I$ $(2 \le i \le n)$ and $C_{ii}(1, j) = 1,\ (1 \le i \le n-1,\ 2 \le j \le n)$. Let $p = \omega(in+1, in+j)$, $q = \omega(j, in+j)$. Using triangles $(1, j, in+j)$ and $(1, in+1, in+j)$ which must have weight $+1$ it is clear that $p = q = 1$. Thus

$$C_{1,i+1}(j,j) = \omega(j, in+j) = 1,\ (1 \le j \le n, 1 \le i \le n-1).$$

Also, $p = 1$ implies

$$C_{i+1,i+1}(1, j) = \omega(in+1, in+j) = 1,\ (2 \le j \le n).$$

Step 2. Show that $D = 2A$. Consider triangles $(1, z, in+2)$ of type $(2,1,2)$. Using Step 1 we have, for triangles $(2, x, in+2)$ and $(in+1, y, in+2)$ of type

$(1,1,1)$,

$$\begin{aligned} d &= \left|\{z \mid \delta\omega(1,z,in+2)=1\}\right| \\ &= \left|\{x \mid \omega(x,in+2)=1\}\right| + \left|\{y \mid \omega(in+2,y)=1\}\right| \\ &= \left|\{x \mid \delta\omega(2,x,in+2)=1\}\right| + \left|\{y \mid \delta\omega(in+1,y,in+2)=1\}\right| \\ &= 2a. \end{aligned}$$

So $D = 2d - 2(n-2) = 4a - 2n + 4 = 2(2a-n+2) = 2A$.

Step 3. Show $C_{ii} = C_{11}$ $(1 \le i \le n)$, $C_{ij} = \pm I$, $(i \ne j)$. By Step 1,

$$\omega(j, in+j) = \omega(k, in+k) = 1.$$

By RP applied to rectangle $(in+j, j, k, in+k)$,

$$\omega(j,k) = \omega(in+j, in+k),\ (0 \le i \le n-1).$$

Thus $C_{ii} = C_{11}$ $(1 \le i \le n)$. Again by Step 1,

$$\omega(in+1, in+k) = \omega(jn+1, jn+k) = 1,$$

so

$$\mathrm{R}P \implies \omega(in+1, jn+1) = \omega(in+k, jn+k),\ (0 \le i \ne j \le n-1).$$

Hence $C_{ij}(1,1) = C_{ij}(k,k)$ $(1 \le k \le n)$, and we have $C_{ij} = \pm I$, $(i \ne j)$.

Step 4. Show that $F = A^2$. Consider the $(n-2)^2$ triangles $(1, in+j, n+2)$ of type $(2,2,2)$. Let $p = \omega(in+2, in+j)$, $q = \omega(n+2, in+2)$ and $s = \omega(n+2, in+j)$. Using step 2,

$$q = 1 \iff \delta\omega(1, in+1, n+1) = 1.$$

Thus $\left|\{i \mid \omega(n+2, in+2) = 1\}\right| = a$. Likewise,

$$p = 1 \iff \delta\omega(1,j,2) = 1.$$

Hence $\left|\{j \mid \omega(in+2, in+j) = 1\}\right| = a$. Since $\delta\omega(in+2, n+2, in+j) = 1$, $s = 1 \iff q = p$. Therefore,

$$\begin{aligned} f &= \left|\{in+j \mid \delta\omega(1, in+j, n+2) = 1\}\right| \\ &= \left|\{\{i,j\} \mid q = p = 1\}\right| \quad + \quad \left|\{\{i,j\} \mid q = p = -1\}\right| \\ &= a^2 + (n-2-a)^2 \\ &= 2a^2 - 2a(n-2) + (n-2)^2. \end{aligned}$$

Now $F = 2f - (n-2)^2 = (2a - (n-2))^2 = A^2$.

Define the matrix Q by

$$Q(i,j) = C_{ij}(1,1) \ (i \neq j),$$
$$Q(i,i) = 0.$$

Set $P = C_{11}$, and observe that $P(i,i) = 0$ for $1 \leq i \leq n$. We have shown that

$$C_1^\omega = I \otimes P + Q \otimes I.$$

Step 5. Show $C_2^\omega = Q \otimes P$. By Step 2, we have

$$\omega(in+k, in+l) = \omega(jn+k, jn+l) = P(k,l)$$
$$\omega(in+k, jn+k) = \omega(in+l, jn+l) = Q(i+1, j+1).$$

Now, triangle $(in+k, in+l, jn+l)$ has type $(1,1,2)$, so

$$\omega(in+k, jn+l) = P(k,l)Q(i+1, j+1).$$

Thus $\overline{C}_{ij}(k,l) = P(k,l)Q(i,j) \implies \overline{C}_{ij} = Q(i,j)P$. Hence $C_2^\omega = Q \otimes P$.

It remains to show that $\omega_1 := I + Q$ and $\omega_2 := I + P$ are matrices of regular 2–graphs with identical parameters. From M_1^ω we know that

$$(C_1^\omega)^2 = 2(n-1)I + A(C_1^\omega) + 2(C_2^\omega).$$

Thus

$$I \otimes P^2 + Q^2 \otimes I + 2Q \otimes P = 2(n-1)I_{n^2} + AI \otimes P + AQ \otimes I + 2Q \otimes P$$

which implies that

$$P^2 = (n-1)I + AP,$$
$$Q^2 = (n-1)I + AQ.$$

By Taylor's theorem, P and Q are matrices of regular 2–graphs, and it follows that n must be even. Both regular 2–graphs have the same equation, hence the same parameters. We now have

$$\omega = I + C_1^\omega + C_2^\omega = \omega_1 \otimes \omega_2,$$

proving the theorem. $\square$

From steps 2 and 4 above we know the weighted intersection matrices associated with ω on Γ are

$$M_1^\omega = \begin{pmatrix} 0 & 1 & 0 \\ 2(n-1) & A & 2 \\ 0 & n-1 & 2A \end{pmatrix} \quad M_2^\omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & n-1 & 2A \\ (n-1)^2 & (n-1)A & A^2 \end{pmatrix}.$$

Analyzing eigenvalues and multiplicities, there are two possibilities:

(i) $A^2 + 4(n-1)$ is not a square. Then $A = 0$, the eigenvalues of C_1^ω are 0, $\pm 2\sqrt{n-1}$ with multiplicities $\frac{n^2}{2}, \frac{n^2}{4}, \frac{n^2}{4}$. Here $\delta\omega_1$ and $\delta\omega_2$ are conference 2-graphs of order n. Necessary conditions for the existence of conference 2-graphs are: (1) $n \equiv 2 \pmod 4$ and (2) $n-1$ is the sum of two squares ([13], [15]). The regular 2–graph associated with the example on $L_2(6)$ discussed earlier is of this type.

(ii) $A^2+4(n-1) = s^2$, for some positive integer s. Then eigenvalues are $A, A\pm s$ with multiplicities

$$m_1 = \frac{2(n-1)n^2}{s^2}, \quad m_2 = \frac{n^2(s^2 - 2(n-1) - As)}{2s^2},$$

$$m_3 = \frac{n^2(s^2 - 2(n-1) + As)}{2s^2}.$$

Note that $A = n-2, s = n$ is always a solution, and it affords the trivial weight $\omega \equiv 1$. Other examples of this type include the unique regular 2-graphs with $(n, A, s) = (16, 2, 8)$ and $(n, A, s) = (28, 6, 12)$. (See [15] for more information about existence.)

Remark. $\delta\omega$ is a regular 2-graph $\iff A + 2(n-1) + (n-1)A = 2 + 4A + A^2 \iff A = -2$ or $A = n-2$. If $A = n-2$ then both $\delta\omega_1$ and $\delta\omega_2$ are trivial 2-graphs, hence $\delta\omega$ is trivial. If $A = -2$ then $a = \frac{n-4}{2}$ and n are the parameters of $\delta\omega_i$ $(i = 1, 2)$, while $\frac{n^2-4}{2}$ and n^2 are the parameters of $\delta\omega$.

5. Discussion

In the group case, with notation as in section 3, the following three statements are equivalent: (i) ω is trivial; (ii) λ^G has one constituent of degree 1; (iii) G has an index 2 subgroup N such that $N \cap H = A$ ([10]). Condition (iii) is the reason why we use the group $2.L_3(4)$ in that example, rather than $L_3(4).2$, when searching for a non-trivial regular weight.

Feasible parameters have been generated for regular weights of rank 2 or 3 on all srg's with $n \leq 400$ ([10]). For most of these parameter sets, existence of a regular weight is unknown. No example of a non-trivial regular rank 3 weight on a Type I srg ([1,3]) is known.

References

1. A.E. Brouwer and J.H. van Lint, *Strongly regular graphs and partial geometries*, Enumeration and Design, Academic Press Canada, Toronto and Orlando, 1984, pp. 85–122.
2. J. Cannon, *An introduction to the group theory language, Cayley*, Computational Group Theory (Durham, 1982), Academic Press, London and Orlando, 1984, pp. 145–183.
3. P.J. Cameron and J.H. van Lint, *Designs, Graphs, Codes and their Links*, London Math. Soc. Student Texts, vol. 22, Cambridge University Press, Cambridge and New York, 1991.
4. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford and New York, 1985.
5. P. Delsarte, J.M. Goethals and J.J. Seidel, *Bounds for systems of lines, and Jacobi polynomials*, Philips Res. Rep. **30** (1975), 91–105.
6. M. Hall, R. Lane and D. Wales, *Designs derived from permutation groups.*, J. Combin. Theory **8** (1970), 12–22.
7. D.G. Higman, *Coherent algebras*, Linear Algebra Appl. **93** (1987), 209–239.
8. ———, *Weights and t-graphs*, Bull. Soc. Math. Belg. **XLII** (1990), 501–521.
9. J. H. Lindsey II, *On a six dimensional projective representation of* $PSU_4(3)$, Pacific J. Math. **36** (1971), 407–425.
10. A. D. Sankey, *Regular weights on strongly regular graphs*, Ph.D. thesis, University of Michigan, 1992.
11. W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs, 1964.
12. J. J. Seidel, *Polytopes and non-Euclidean geometry*, Mitt. Math. Sem. Giessen **163** (1984), 1–17.
13. ———, *A survey of two-graphs*, Proc. Intern. Colloqu. Teorie Combinatorie (Rome 1973), Tomo I Accad. Naz. Lincei (1976), 481–511.
14. J.J. Seidel and D.E. Taylor, *Two-graphs, a second survey*, Algebraic Methods in Graph theory, Szeged (Hungary 1978), North-Holland, Amsterdam and New York, 1981, pp. 689–711.
15. D.E. Taylor, *Regular twographs*, Proc. London Math. Soc. (3) **35** (1977), 257–274.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

Current address: Department of Mathematics, Slippery Rock University, Slippery Rock, PA 16057

E-mail address: ads@sruvm.sru.edu