Overview of research area: association schemes

Three closely related areas:

•Designs – experimental designs in statistics, especially t-designs

•Codes – information transmission

•Graphs – general modelling of networks

Results and methods of one area are applied to another.

Strongly regular graphs (srg's)

•Symmetry inherent in t-designs is reflected in srg's.

•Interaction between these is fruitful.

•Can be represented by matrices with nice algebraic properties.

An example would help

Example to illustrate the interplay between designs and srg's.

•Start with a pentagon, an srg.

•Construct from this a $2-(n,3,\lambda)$ design.

•Construct the "block graph" of the design.

•The graph is a (different) srg.

Construct the design

•The *blocks* of the design are sets of 3.

•Choose these according to criterion: number of connections in pentagon graph is odd.

All possible sets of 3:

123	134	145	156	345
124	135	146	245	346
125	136	235	246	356
126	234	236	256	456

Sets of 3 that satisfy the condition:

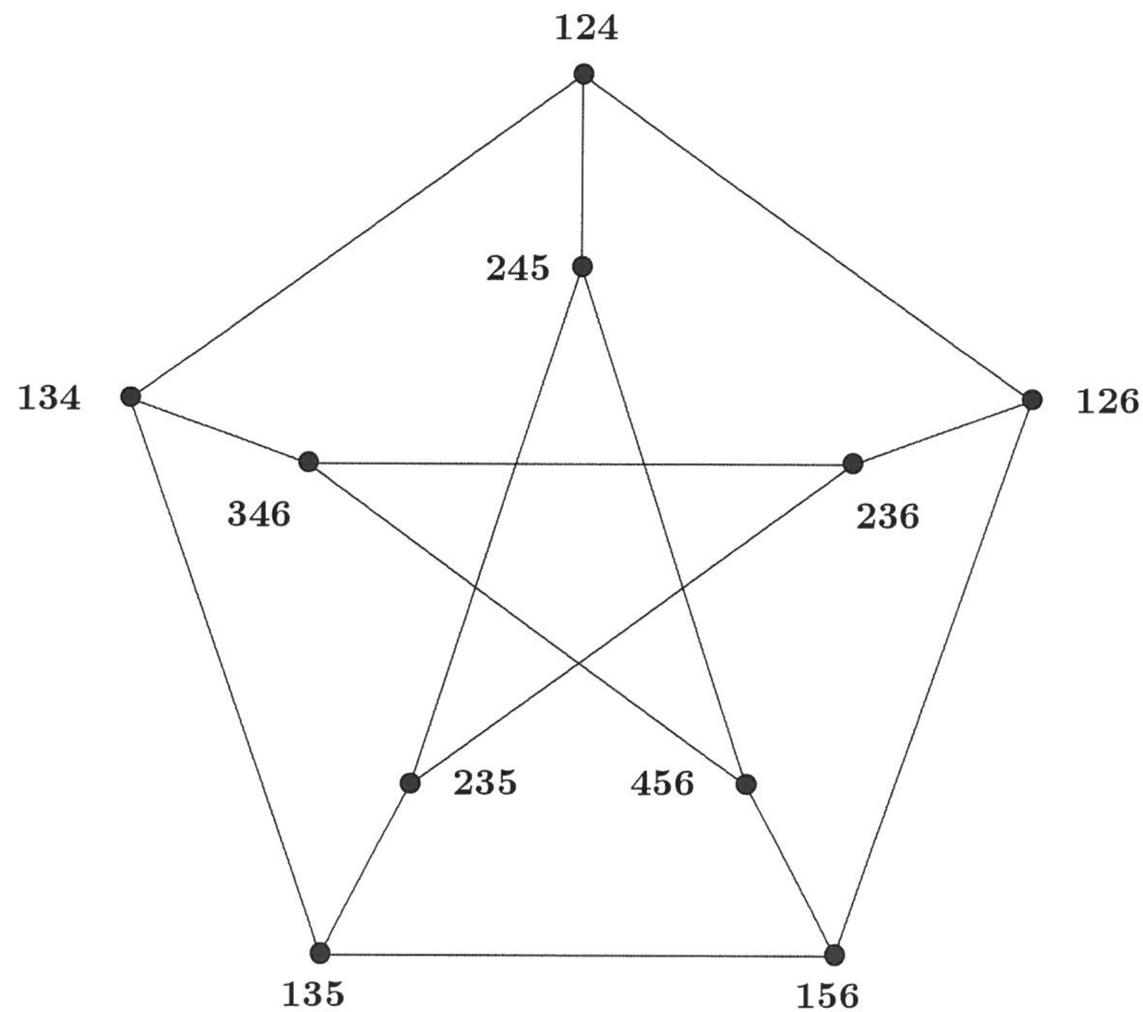
124	134	235	156	346
126	135	236	245	456

These sets are the blocks of a $2-(6,3,2)$ design.

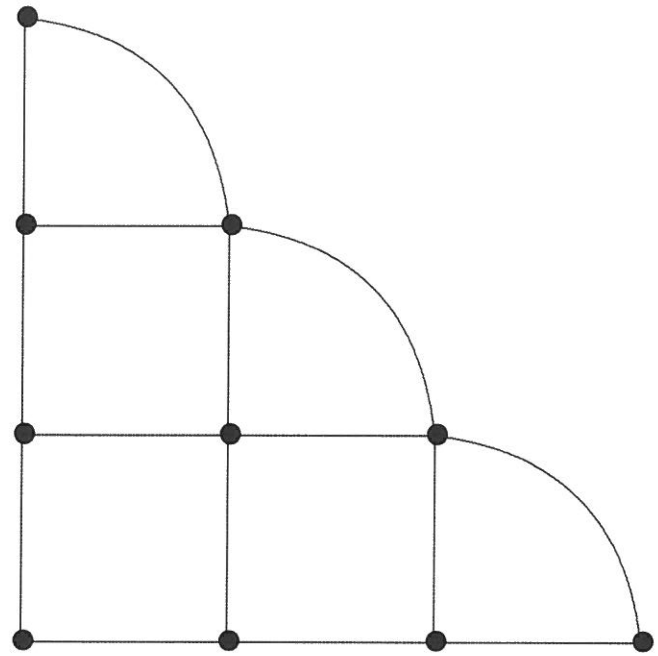
Points	Blocks 1	2	3	4	5	6	7	8	9	10
1	x	x	x	x			x			
2	x	x			x	x		x		
3			x	x	x	x			x	
4	x		x					x	x	x
5				x	x		x	x		x
6		x				x	x		x	x

Construct the block graph

- Make the blocks into points of a new graph.
- Two are connected if they intersect in 2 points.
- This graph is strongly regular!
- Its sibling is known as the Triangular Graph, below.



124
245
134
126
346
236
235
456
135
156



Where do the association schemes come in?

- The two complementary block graphs above, along with the "equals" graph, form an association scheme.
- srg's are the smallest *rank* association schemes.
- Studied in the 1950's by Bose, Shimamoto, and others.
- Important in group theory in the 1970's. Tie in to coding theory here.
- Delsarte in '73 united the sphere-packing bound (major result in coding) and Fisher's inequality (major result in designs) using association schemes.
- Result shows association schemes to be the basic underlying structure of both.

So what is an association scheme?

•Set of points, X and some number of (symmetric) relations.

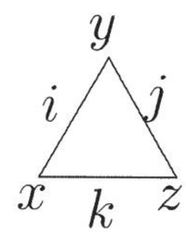
•Think of each relation as a graph.

•Three properties are essential:

1. Every pair of points is joined in (exactly) one of the graphs.
2. One of the relations is equality.

Third property requires some terminology.

Given points x and z that are joined in graph k, count triangles:



Here, everything is fixed except y.

Call this number $p_{ij}^{k}(x,z)$. The third property is:

3. $p_{ij}^{k}(x,z)$ is independent of the choice of x and z.

Call these numbers p_{ij}^{k}. They are the *parameters* of the association scheme.

Equivalent algebraic formulation

Let A_i = matrix of the i^{th} relation, or (equivalently) the adjacency matrix of the graph. These are square matrices, with rows and columns indexed by the points, and (x, y) entry equalling 1 or 0, according as x and y are i^{th} associates or not.

$$
\begin{aligned}
&(i) \qquad && A_0 = I \\
&(ii) \qquad && \sum A_i = J \\
&(iii) \qquad && A_i \cdot A_j = \sum_k p_{ij}^k A_k
\end{aligned}
$$

Importance of the matrix formulation:

- The linear span of the matrices is closed under multiplication.
- They form a basis of an associative algebra called the Bose-Mesner algebra.

Weighted association schemes

Weight the edges in each graph so that:

•Something new (interesting?)is obtained.

•Nice algebraic properties are retained.

Main result

•Put a weight as above on a known family of srg's.

•Use values ± 1.

•Determine when this is possible, and what the resulting structures are.

The following theorem shows that all non-trivial regular rank 3 weights on $L_2(n)$ are obtained from 2–designs.

Theorem If ω is a non-trivial regular weight with values ± 1 and full support on the lattice graph $L_2(n)$ then n is even and $\omega = \omega_1 \otimes \omega_2$, where $\delta\omega_1$ and $\delta\omega_2$ are regular 2–graphs with the same parameters.

Note: A regular 2–graph is a $2-(n,3,\lambda)$ design.

Example of a regular weight

Let Γ be the lattice graph $L_2(6)$. Let $A_1,\ A_2$ be adjacency matrices of Γ and $\overline{\Gamma}$ respectively. Put

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

C is a conference matrix of order 6 ($C^2 = 5I$), hence represents a regular 2–graph on 6 vertices. Put

$$\begin{aligned} \omega &= (I + C) \otimes (I + C) \\ &= I \otimes I + I \otimes C + C \otimes I + C \otimes C. \end{aligned}$$