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On t-fold covers of coherent configurations

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Abstract

We introduce the covering configuration induced by a regular weight defined on a coherent configuration. This construction generalizes the well-known equivalence of regular two-graphs and antipodal double covers of complete graphs. It also recovers, as special cases, the rank 6 association schemes connected with regular 3-graphs, and certain extended Q-bipartite doubles of cometric association schemes. We articulate sufficient conditions on the parameters of a coherent configuration for it to arise as a covering configuration.

Keywords: Association scheme, coherent configuration, regular weight, double cover, two-graph, t-graph
Math. Subj. Class.: 05C22, 05C50, 05E30

1 Introduction

The Seidel matrix of a graph Γ may be viewed as a weight on the complete graph: edges of Γ are weighted (-1) and non-edges $(+1)$. If Γ is strongly regular with $n = 2(2k - \lambda - \mu)$, it lies in the switching class of a regular two-graph and we call the weight, analogously, *regular* on K_n. This condition on Γ is well known, and dates to 1977, in [25]. The same year, the equivalence of regular two-graphs and antipodal double covers of complete graphs was established in [26].

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Martin, Muzychuk and Williford ([18]) defined the *extended Q-bipartite double* of a cometric association scheme, extending the notion of the bipartite double of a distance regular graph. This construction produces, as special cases, the antipodal double covers of complete graphs from the strongly regular graphs affording regular two-graphs.

In recent work, Kalmanovich has also generalized the regular two-graph result, working from an unpublished draft of D. G. Higman's ([16], [9]) on *regular 3-graphs*. As defined in [14], a t-graph weights the edges of K_n with elements of the group of roots of unity of order t, U_t. The regularity condition ensures that the matrix of edge weights has a quadratic minimal polynomial. The work of Kalmanovich-Higman establishes the equivalence of regular 3-graphs with cyclic antipodal 3-fold covers of K_n ([6]). Regular 3-graphs are shown to give rise to certain rank 6 association schemes, and the necessary conditions under which a rank 6 scheme arises in this way are given.

In this paper there are two main results. First, working with a regular weight with values in U_t, defined on a coherent configuration (CC), we show that there is always a *covering configuration*; that is, a CC constructed using a t-fold cover in a natural way, to convert the weight into a CC of higher rank (by a factor of t). As special cases, we recover the equivalence between regular two-graphs and antipodal double covers of complete graphs; some extended Q-bipartite doubles of cometric schemes; the rank 6 schemes associated with regular 3-graphs, and an extension of these to regular t-graphs.

A CC with a regular weight has two sets of parameters: the structure constants for the weighted adjacency algebra, $\{\beta_{ij}^k\}$, which lie in $\mathbb{C}$ or more specifically in the ring of integers with a primitive t^{th} root of unity adjoined, and the non-negative integers $\{\beta_{ij}^k(\nu)\}$ which count certain triangles with a specified weight. They are related by

$$\beta_{ij}^k = \sum_{\nu \in U_t} \nu \beta_{ij}^k(\nu).$$

The weighted adjacency algebra is in general not a coherent algebra, and may in fact have a coherent closure that is much higher in rank than the original CC. In the regular two-graph case, for instance, it is precisely when the (-1) edges form an SRG that we get a minimal closure: a natural fission of the edge set into $(+1)$ and (-1) edges that yields a (rank 3) association scheme. The covering configuration is the realization of a CC whose structure constants are the $\beta_{ij}^k(\nu)$. Some properties, namely homogeneity and commutativity of a CC carry over to the covering configuration. Symmetry is preserved only if $t = 2$. Metric and cometric properties are not.

The second main result of this paper is the articulation of sufficient conditions for a CC to be the covering configuration of a regular weight.

In the final section, we describe a family of regular weights on the Hamming Scheme $H(n, 2)$ with values in U_4, due to Ada Chan. These weights all fuse to regular 4-graphs, providing an infinite family that may be of interest as complex Hadamard matrices. These regular weights, and their fusions, admit covering configurations of ranks $4(n + 1)$ and 8 respectively, on 2^{n+2} points.

2 Preliminaries

In this section, we give the definitions that are essential to what follows. Much more can be found in [17] and in the original developments of the area by Weisfeiler and Lehman in [28] and by D.G. Higman in [11], [12], and [14].

2.1 Coherent configurations

Definition 2.1. Let $\{A_i\}_{0\leq i<r}$ be a set of 01-matrices with rows and columns indexed by a finite set X. Let $\mathcal{I} := \{0, 1, \ldots, r\}$. The linear span $\mathcal{A} := \langle A_i\rangle_{\mathbb{C}}$ is a *coherent algebra* if:

(i) $\sum_{i\in\mathcal{I}} A_i = J$, where J is the all-ones matrix,

(ii) $\sum_{i\in\mathcal{L}} A_i = I$, for some subset $\mathcal{L} \subset \mathcal{I}$,

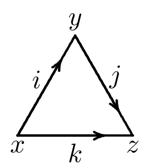
(iii) for each i there exists $i^* \in \mathcal{I}$ such that $A_i^T = A_{i^*}$,

(iv) $A_iA_j = \sum p_{ij}^k A_k$, $p_{ij}^k \in \mathbb{Z}^+$.

A coherent algebra (CA), is *homogeneous* if $|\mathcal{L}| = 1$; *symmetric* if $i^* = i$ for all i, and *commutative*, clearly, if $p_{ij}^k = p_{ji}^k$ for all i, j, k. The homogeneous CAs are (possibly non-symmetric) association schemes. Commutative schemes which have the *metric* or *P-polynomial* property are synonymous with *distance-regular graphs* (DRGs); those of diameter 2 are the *strongly regular graphs* (SRGs). Some familiarity with these structures is assumed. References for readers lacking this background are [1], [4], [2], [27], [5], and [19]. In the association scheme literature, a rank r scheme is often referred to as an $(r-1)$-class scheme: 'rank' counts the trivial relation, while the number of 'classes' does not.

Every algebra of n by n matrices over $\mathbb{C}$ that is closed under transpose and entry-wise multiplication, and contains both I and J is a coherent algebra, and as such it has a basis of 01-matrices satisfying (i)–(iv). Each A_i in a CA is the adjacency matrix of a digraph Γ_i with vertex set X, which is simple for $i \notin \mathcal{L}$ and a graph when $i^* = i$. Viewing these graphs as relations on X, define a *coherent configuration* (CC) to be a set of binary relations on X, indexed by $\mathcal{I}$, with analogous properties to (i)–(iv) above. Denote it $\mathfrak{A} := (X, \{R_i\}_{i\in\mathcal{I}})$.

The constant p_{ij}^k counts the number of i-j paths from a vertex x to a vertex z, given that $(x, z) \in R_k$ and this number is necessarily independent of the choice of edge in Γ_k. It is convenient to denote each instance of an i-j path by a *triangle* (x, y, z) *of type* (i, j, k). That is, $(x, y, z) \in X^3$ is a triangle of type (i, j, k) if $(x, y) \in R_i$, $(y, z) \in R_j$, and $(x, z) \in R_k$ as indicated in the figure.



Define the *intersection matrices* M_j of a CC by $M_j := \left(p_{ij}^k\right), 0 \leq i, k < r$ thus the map

$$\gamma : A_j \mapsto M_j$$

is the right regular representation of $\mathcal{A}$.

We treat CAs and CCs as equivalent structures and move freely between the notations of matrices, relations, and graphs. As $\{A_i\}$ forms the *standard basis* of $\mathcal{A}$, we refer to $\{R_i\}$ and $\{\Gamma_i\}$ as the *basic relations* and *basic graphs* of $\mathfrak{A}$ respectively.

2.2 Fusion and fission

A *fusion* is a merging of relations in a CC according to a partition of $\mathcal{I}$. A fusion will be deemed *coherent* if the resulting configuration is coherent. A coherent *fission* or *refinement* is a partition of each basic relation such that the resulting set of relations forms a CC.

The rank 2 CC represented by K_n is the minimum element in the lattice of all CCs on a given vertex set X of size n ([12], Prop. 3). The maximum element has rank n^2, with the full matrix algebra $M_X(\mathbb{C})$ as its coherent algebra.

2.3 Regular weights

Let $U = U_t$ be the group of complex t^{th} roots of unity, and fix a primitive root ζ as the generator of U.

Definition 2.2. A *weight* with values in U is a 2-cochain $\omega : X^2 \to U$. Viewed as a matrix, a weight is Hermitian with unit diagonal.

The *coboundary* of ω is a function on triangles:

$$\delta\omega(x, y, z) := \omega(y, z)\overline{\omega(x, z)}\omega(x, y)$$

and we refer to this value as the *weight* of the triangle (x, y, z). Analogous to Seidel switching on a graph, switching a weight ω at vertex x_i by a factor of $\alpha \in U$ multiplies the weight on (x_i, y) edges by α and on (y, x_i) edges by $\bar{\alpha}$ for all $y \neq x_i$. In matrix form, this is a similarity transform by the diagonal matrix $\mathrm{diag}(1, 1, \ldots, 1, \alpha, 1, \ldots, 1)$ with α in position i. We refer to two weights as *switching equivalent* if one is obtained from the other by some sequence of switches, and observe that $\delta\omega$ is invariant under switching.

Definition 2.3. A *t-graph* is $\delta\omega$ for some weight ω. It is *regular* if

$$|\{y \mid \delta\omega(x, y, z) = \alpha\}|$$

is independent of x and z, for each value $\alpha \in U$.

This is one of a number of natural generalizations of the regular two-graph ([25], [22], [24], [23], [9], [16]). Since a 2-cochain is equivalent to a weight on the edges of a complete graph, the notion of regularity can be extended to weights on CAs.

The entry-wise product $\omega \circ A_i$ gives a matrix with (x, y) entry equal to $\omega(x, y)$ where $(x, y) \in R_i$. Denote this *weighted adjacency matrix* A_i^ω.

Definition 2.4. [14] A weight ω is *regular* on a CC if for $(x, z) \in R_k$ the number of triangles (x, y, z) of type (i, j, k) and weight α is independent of x and z. In this case, the number of such triangles depends on i, j, k, and α and we denote this parameter $\beta_{ij}^k(\alpha)$.

If ω is regular on $\mathfrak{A}$, then $\sum_\alpha \beta_{ij}^k(\alpha) = p_{ij}^k$. By a straight-forward counting argument,

$$A_i^\omega A_j^\omega = \sum_k \beta_{ij}^k A_k^\omega \text{ where } \beta_{ij}^k := \sum_{\alpha \in U} \alpha \beta_{ij}^k(\alpha)$$

thus $\mathcal{A}^\omega := \langle A_i^\omega \rangle$ is a self-adjoint matrix algebra containing I and we refer to the β_{ij}^k as the *parameters* or *structure constants* of the $\mathcal{A}^\omega$. Note that this *weighted adjacency algebra* is not necessarily closed under the entry-wise product, hence it is not, in general, a coherent algebra. The *weighted intersection matrices* are defined in the obvious way,

$$M_j := \left(\beta_{ij}^k\right), 0 \le i, k < r.$$

Switching equivalent weights have identical parameters and therefore identical intersection matrices.

2.4 The fission induced by a weight

The weighted CC $(\mathfrak{A}, \omega)$ has a natural fission in which R_i is partitioned according to distinct values of ω. Put

$$(A_i^\alpha)_{xy} := \begin{cases} 1 & \text{if } (A_i^\omega)_{xy} = \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

Some useful properties are:

1. $A_i^\alpha \circ A_j^\beta = \delta_{i,j}\delta_{\alpha,\beta}A_i^\alpha$;
2. $A_i = \sum_\alpha A_i^\alpha$;
3. $A_i^\omega = \sum_\alpha \alpha A_i^\alpha$

Definition 2.5. $(\mathfrak{A}, \omega)$ has *minimal closure* if the fission $\{A_i^\alpha\}$ forms a CC.

The terminology draws on the notion of the *coherent closure* of a set of matrices as the smallest CA containing them (see [28], [21] for more). The coherent closure of $(\mathfrak{A}, \omega)$ is the CC whose CA is the coherent closure of the matrix algebra $\mathcal{A}^\omega$. Clearly

$$\sum_{i \in \mathcal{L}} A_i^1 = I$$

and the A_i^α sum to J. Furthermore, by the Hermitian property of the weight,

$$\left(A_i^\alpha\right)^T = A_{i^*}^{\overline{\alpha}}$$

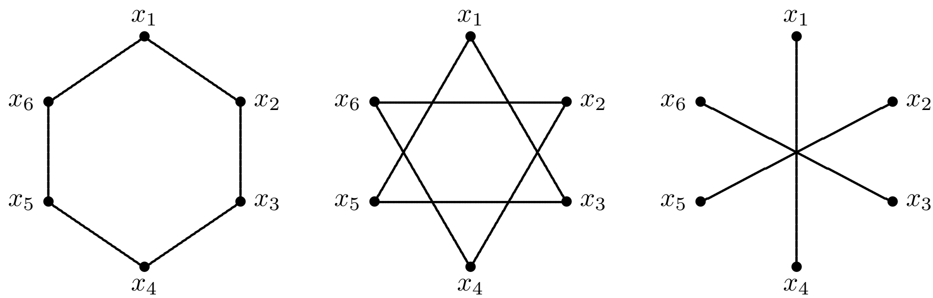
but the fission is not in general coherent and may in particular generate a matrix algebra of dimension greater than rt.

A weighted CC may be represented in a natural way as a t-fold cover of the configuration. The main goal of this work is to characterize regular weights on CCs in this way, and to describe the construction of a CC of rank rt – the *covering configuration* – derived from the cover.

Let $U = U_t$ with generator ζ, let Γ be a graph or digraph with vertex set X, and ω a weight on Γ. Following [14], we define the t-fold cover of Γ *afforded* by ω as follows. The vertex set is $X \times \{1, 2, \ldots, t\}$. We abuse notation, denoting the t copies of each vertex x by $\{x_1, x_2, \ldots, x_t\}$. Assign adjacencies by $x_i \sim y_j$ whenever $x \sim y$ in Γ and $\omega(x, y) = \zeta^{i-j}$. The induced permutation of indices, $i \mapsto j$ determines a permutation σ of U, namely $\zeta^k \mapsto \zeta^{k+j-i}$ which is simply multiplication by ζ^{j-i}. Let Z_σ be the image of

$\sigma \in U$ in the left regular representation of U as a multiplicative group. Then $\{Z_\sigma \mid \sigma \in U\}$ is a cyclic group generated by Z_ζ and the element Z_{ζ^k} corresponds to the k^{th} power of the cycle $(1, 2, \ldots, t)$ on indices. Observe that $\sum_{\sigma \in U} Z_\sigma$ is the all-ones matrix J. Indeed, the Z_σ are the adjacency matrices of a cyclic group scheme on t points.

Example 2.6. We construct a weight with values in U_3 on the cycle C_3, a DRG of diameter 3. The non-trivial basic graphs are shown below.



Define a weight ω by:

$$A_1^\omega = \begin{bmatrix} & \alpha & & & & \bar{\alpha} \\ \bar{\alpha} & & \alpha & & & \\ & \bar{\alpha} & & \alpha & & \\ & & \bar{\alpha} & & \alpha & \\ & & & \bar{\alpha} & & \alpha \\ \alpha & & & & \bar{\alpha} & \end{bmatrix}, \quad A_2^\omega = \begin{bmatrix} & & \bar{\alpha} & & \alpha & \\ & & & \bar{\alpha} & & \alpha \\ \alpha & & & & \bar{\alpha} & \\ & \alpha & & & & \bar{\alpha} \\ \bar{\alpha} & & \alpha & & & \\ & \bar{\alpha} & & \alpha & & \end{bmatrix},$$

$$A_3^\omega = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{bmatrix}.$$

Working out the products, we see that

$$\begin{aligned} (A_1^\omega)^2 &= 2I + A_2^\omega, & (A_2^\omega)^2 &= 2I + A_2^\omega, \\ A_1^\omega A_2^\omega = A_2^\omega A_1^\omega &= A_1^\omega + 2A_3^\omega, & A_2^\omega A_3^\omega = A_3^\omega A_2^\omega &= A_1^\omega, \\ A_1^\omega A_3^\omega = A_3^\omega A_1^\omega &= A_2^\omega, & (A_3^\omega)^2 &= I, \end{aligned}$$

and therefore the weighted intersection matrices are

$$M_1^\omega = \begin{bmatrix} & 1 & & \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ & & 1 & \end{bmatrix}, \quad M_2^\omega = \begin{bmatrix} & & 1 & \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ & 1 & & \end{bmatrix}, \quad M_3^\omega = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Note. Merging the non-trivial relations or, equivalently, summing $A_i, i \neq 0$, and also the A_i^ω, we see that this weight fuses to a regular 3-graph.

3 Main theorem

Theorem 3.1. *Let* $\mathfrak{A} = (X, \{R_i\}_{i\in\mathcal{I}})$ *be a coherent configuration of rank* r *on* $n := |X|$ *vertices and suppose* ω *is a regular weight on* $\mathfrak{A}$ *with values in* $U = U_t$*. Then* ω *induces a rank* tr *coherent configuration on* tn *vertices with relations given by*

$$\sum_{\alpha\in U} A_i^\alpha \otimes Z_{\sigma\alpha} \quad (i \in \mathcal{I}, \sigma \in U)$$

and parameters $\{\beta_{ij}^k(\alpha)\}$.

Proof. Let $T := \{1, 2, \dots, t\}$ and let Γ_i be one of the basic graphs in $\mathfrak{A}$. The t-fold cover of Γ_i that is induced by ω has vertex set $Y := X \times T$, and adjacency matrix

$$\sum_{\alpha\in U} A_i^\alpha \otimes Z_\alpha.$$

Motivated by this, and looking to define the matrices of a CA on Y, we put

$$C_{i,\sigma} := \sum_{\alpha\in U} A_i^\alpha \otimes Z_{\sigma\alpha}, \tag{3.1}$$

for $i \in \mathcal{I}$ and $\sigma \in U$, we claim that $\mathcal{C} := \langle C_{i,\sigma}\rangle_{\mathbb{C}}$ is the coherent algebra of a CC $\mathfrak{C}$.

We show that $\mathcal{C}$ satisfies (i)–(iv) of Definition 2.1. We have observed that $\sum_{\sigma\in U} Z_\sigma = J$. Since $\sum_{\alpha\in U} A_i^\alpha = A_i$ for all i, and $\sum_{i\in\mathcal{I}} A_i = J$, we see that

$$\begin{aligned}
\sum_{i\in\mathcal{I}}\sum_{\sigma\in U} C_{i,\sigma} &= \sum_{i\in\mathcal{I}}\sum_{\sigma\in U}\sum_{\alpha\in U} A_i^\alpha \otimes Z_{\sigma\alpha} \\
&= \left(\sum_{i\in\mathcal{I}}\sum_{\alpha\in U} A_i^\alpha\right) \otimes \left(\sum_{\sigma\in U} Z_{\sigma\alpha}\right) \\
&= \left(\sum_{i\in\mathcal{I}} A_i\right) \otimes J_t \\
&= J_n \otimes J_t \\
&= J_{nt}.
\end{aligned} \tag{3.2}$$

Hence $\mathcal{C}$ satisfies (i).

Let $\mathcal{L} \subseteq \mathcal{I}$ be the unique set of indices such that $\sum_{i\in\mathcal{L}} A_i = I_n$. (Assume, without loss of generality, that $\mathcal{L} = \{0\}$ if $\mathcal{A}$ is homogeneous.) We claim $I_{nt} = \sum_{i\in\mathcal{L}} C_{i,1}$. Since $\omega(x, x) = 1$ for all x, $A_i^\alpha = 0$ if $i \in \mathcal{L}$ and $\alpha \neq 1$. Consequently, $i \in \mathcal{L}$ implies $A_i = A_i^1$.

Hence,

$$\begin{aligned}\sum_{i\in\mathcal{L}} C_{i,1} &= \sum_{i\in\mathcal{L}}\sum_{\alpha} A_i^{\alpha}\otimes Z_{\alpha}\\ &= \sum_{i\in\mathcal{L}} A_i\otimes Z_0\\ &= \left(\sum_{i\in\mathcal{L}} A_i\right)\otimes I_t\\ &= I_n\otimes I_t = I_{nt}\end{aligned} \tag{3.3}$$

This proves that $\mathcal{C}$ satisfies (ii).

Remark 3.2. If $\mathfrak{A}$ is an association scheme, then $\mathcal{L}=\{0\}$, and $C_{0,1}=I_{nt}$.

The transpose of $M\otimes N$ is $M^T\otimes N^T$. Since $\omega(y,x)=\overline{\omega(x,y)}$, $(A_i^{\alpha})^T=(A_{i^*})^{\overline{\alpha}}$. The transpose of a permutation matrix is its matrix inverse, hence $Z_\sigma^T=Z_\sigma^{-1}=Z_{\sigma^{-1}}$. Therefore,

$$C_{i,\sigma}^T=\sum_{\alpha} A_{i^*}^{\overline{\alpha}}\otimes Z_{(\sigma\alpha)^{-1}}=\sum_{\overline{\alpha}} A_{i^*}^{\overline{\alpha}}\otimes Z_{\overline{\sigma}\,\overline{\alpha}}=C_{i^*,\overline{\sigma}}$$

thus $\mathcal{C}$ satisfies (iii).

Finally, we obtain the structure constants as follows. We claim:

$$\left(\sum_{\alpha\in U} A_i^{\alpha}\otimes Z_{\sigma\alpha}\right)\left(\sum_{\beta\in U} A_j^{\beta}\otimes Z_{\tau\beta}\right)=\sum_{k\in\mathcal{I}}\sum_{\nu\in U}\beta_{ij}^k(\nu)\sum_{\gamma\in U}\left(A_k^{\gamma}\otimes Z_{\sigma\tau\nu\gamma}\right). \tag{3.4}$$

The left hand side of equation (3.4) is equal to

$$\begin{aligned}\sum_{\alpha,\beta\in U}\left(A_i^{\alpha}A_j^{\beta}\right)\otimes(Z_{\sigma\alpha}Z_{\tau\beta}) &= \sum_{\alpha,\beta\in U} A_i^{\alpha}A_j^{\beta}\otimes Z_{\sigma\tau\alpha\beta}\\ &= \sum_{\mu\in U}\left(\sum_{\alpha\beta=\mu} A_i^{\alpha}A_j^{\beta}\right)\otimes Z_{\sigma\tau\mu},\end{aligned}$$

combining terms with the same second tensorand. We now consider the (x,z) entry of each product $A_i^{\alpha}A_j^{\beta}$ for a fixed $(x,z)\in R_k$, setting $\gamma:=\omega(x,z)$. This equals the number of triangles (x,y,z) of type (i,j,k) with weight $\alpha\beta\overline{\gamma}$. Since we are summing these products over all α and β with $\alpha\beta=\mu$, we account for all such triangles, and the number of these is $\beta_{ij}^k(\alpha\beta\overline{\gamma})$. Thus

$$\begin{aligned}\sum_{\mu\in U}\left(\sum_{\alpha\beta=\mu} A_i^{\alpha}A_j^{\beta}\right)\otimes Z_{\sigma\tau\mu} &= \sum_{\mu\in U}\left(\sum_{\gamma\in U}\sum_{k\in\mathcal{I}}\beta_{ij}^k(\mu\overline{\gamma})A_k^{\gamma}\right)\otimes Z_{\sigma\tau\mu}\\ &= \sum_{\mu\in U}\sum_{\gamma\in U}\sum_{k\in\mathcal{I}}\beta_{ij}^k(\mu\overline{\gamma})\left(A_k^{\gamma}\otimes Z_{\sigma\tau\mu}\right).\end{aligned} \tag{3.5}$$

Next, observe that $\beta_{ij}^k(\nu)$ occurs exactly t times, once for each γ with $\mu = \nu\gamma$. Factoring gives

$$\sum_{k\in\mathcal{I}}\sum_{\nu\in U}\beta_{ij}^k(\nu)\sum_{\gamma\in U}A_k^\gamma\otimes Z_{\sigma\tau\nu\gamma} \tag{3.6}$$

which proves the claim. Hence $\beta_{ij}^k(\nu)$ is the coefficient of $C_{k,\sigma\tau\nu}$ in the product $C_{i,\sigma}C_{j,\tau}$. □

Remark 3.3. The following are clear from the proof of Theorem 3.1.

(i) $\mathfrak{C}$ is homogeneous if and only if $\mathfrak{A}$ is homogeneous.

(ii) $\mathfrak{C}$ is symmetric if and only if $\mathfrak{A}$ is symmetric and ω is real-valued, that is, $t = 2$. C_{i1} is always symmetric if A_i is.

(iii) $\mathfrak{C}$ is commutative if and only if $\mathfrak{A}$ is commutative.

(iv) If the A_i^α form a CC, then we are in the case of minimal closure, and $\mathfrak{C}$ is a fusion of a Kronecker product configuration.

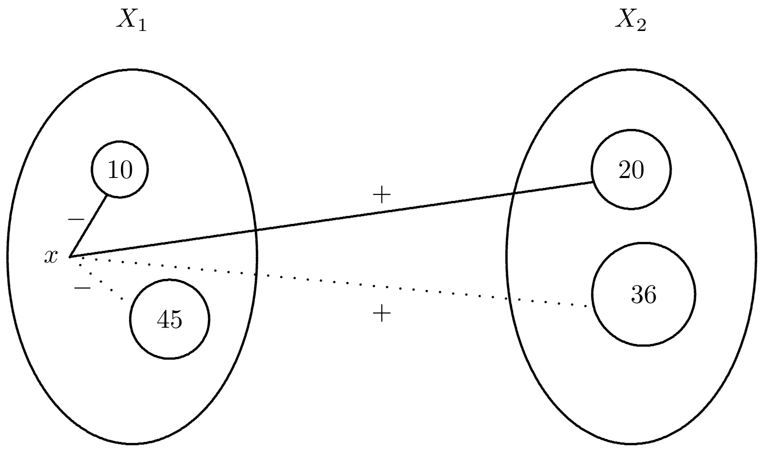
(v) The parameter $\beta_{ij}^k(\nu)$ in the proof of (iv) clearly does not depend on σ or τ. This means that each parameter of $\mathfrak{C}$ is duplicated t^2 times:

$$p_{i\sigma,j\tau}^{k\sigma\tau\nu} = p_{i1,j1}^{k\nu} \quad \forall\sigma,\tau\in U_t.$$

4 Discussion and analysis

Example 4.1. This example relates to $\Gamma = \mathrm{SRG}(112, 30, 2, 10)$ which is known by many names in the literature, including the collinearity graph of $\mathrm{GQ}(3, 9)$, $O^-(6, 3)$, and the first sub-constituent of the McLaughlin graph, McL_1 to name just three. It has a *strongly regular decomposition* into two Gewirtz graphs (SRG(56,10,0,2)) [7].

Let $\mathfrak{A}$ be the rank 3 scheme afforded by Γ. We construct a regular weight on $\mathfrak{A}$ with values in U_2, making use of the decomposition. Let X_1 and X_2 be the two sets of 56 vertices. Define $\omega(x, y)$ for $x \neq y$ to be -1 when x and y are in the same half of this partition, and $+1$ otherwise. Note that ω restricted to either Gewirtz graph is a trivial weight with matrix $2I - J$.



In the figure, a solid line indicates adjacency in Γ, a dotted line non-adjacency. This weighted SRG has minimal closure, since the A_i^α form a rank 5 scheme, in fact a *strongly regular design* or SRD ([13]). Since $\omega(x, y)$ is determined by the parity of $\{x, y\} \cap X_1$, the four non-trivial relations are given by the four combinations of attributes: adjacency/non-adjacency, and this parity. There are many related configurations. For example, another copy of the Gewirtz graph may be adjoined to construct an example of *triality* ([15]). These 112 vertices form the first subconstituent of the McLaughlin graph; the second subconstituent also admits a strongly regular decomposition ([3]).

Some interesting properties of this example:

1. Minimal closure is rare (see [21]).
2. The SRD is cometric, but not metric, which is also rare.
3. The covering configuration $\mathfrak{C}$ is also cometric, but not metric, having rank 6 on 224 points. This example arises as the Q-bipartite double of McL_1 (see [18]).

The Gewirtz graph admits a non-trivial regular weight with values in U_4, constructed via a monomial representation of $2.\mathrm{L}_3(4)$ ([20]). The covering configuration is neither metric nor cometric, has rank 12 on 224 vertices, and contains the doubled Gewirtz graph (DRG[10,9,8,2,1;1,2,8,9,10]) as a quotient.

4.1 Intersection matrices

Lemma 4.2. *The intersection matrices of $\mathcal{C}$ have the form* $M_{j\tau} = \sum_{\nu \in U_t} M_j^\nu \otimes Z_{\tau\nu}$ *where*

$\left[M_j^\nu\right]_{ik} := \beta_{ij}^k(\nu).$

Proof. We may assume the relations $C_{i\sigma}$ are ordered lexicographically, that is first by i and then by $\sigma \in \{1, \zeta, \dots, \zeta^{t-1}\}$ so that the intersection matrix $M_{j\tau}$ has (i,k) block given by $\left(p^{k\sigma\tau\nu}_{i\sigma,j\tau}\right)_{\sigma,\sigma\tau\nu}$. By equation (3.6) this block has the value $\beta^k_{ij}(\nu)$ in position $(\sigma, \sigma\tau\nu)$ which means that it has the form $\sum_\nu \beta^k_{ij}(\nu) Z_{\tau\nu}$. Hence $M_{j\tau}$ is the required sum of Kronecker products. □

Lemma 4.3. *Let ω be a regular weight on the cc $\mathfrak{A}$, and let $\widetilde{\omega}$ be an equivalent weight obtained by switching ω by a factor of $\tau = \zeta^l$ at vertex x. Further let $\mathfrak{C} = (Y, \{C_{i\sigma}\})$ and $\widetilde{\mathfrak{C}} = (Y, \{\widetilde{C}_{i\sigma}\})$ be the covering configurations induced by ω and $\widetilde{\omega}$ respectively. Then $\widetilde{\mathfrak{C}}$ is obtained from $\mathfrak{C}$ by permuting $\{x_i\}$ according to the permutation $(1, 2, \dots, t)^l$ resulting from multiplication by τ on U.*

Proof. Suppose $\omega(x, y) = \alpha$. For some i and j, $(x_1, y_j) \in C_{i1}$ of $\mathcal{C}$, thus $\alpha = \zeta^{j-1}$. Now, $\widetilde{\omega}(x, y) = \tau\alpha$ by assumption, so we have $(x_{1-l}, y_j) \in \widetilde{C}_{i1}$. But this implies that $\widetilde{\mathfrak{C}}$ is obtained from $\mathfrak{C}$ by the permutation $x \mapsto x_{1-l}$, which corresponds to multiplication by τ on U. □

4.2 Special cases

- If ω has minimal closure, $\mathfrak{C}$ is a fusion of a tensor product of two CCs.
- If ω is trivial in the sense that $A^\alpha_i = 0$ for all but one value of α, ω has minimal closure, and $\mathcal{C} = \mathcal{A}^\omega \otimes Z$.
- If $\mathfrak{A}$ has rank 2 (ω is regular on K_n), $\mathfrak{C}$ is a t-fold cover of K_n. It is not necessarily distance regular. This case encompasses the regular two-graphs ($t = 2$), and the regular 3-graphs ($t = 3$) of Higman [9] and Kalmanovich [16].
- If $t = 2$, $\mathfrak{A}$ is a (symmetric) scheme, and $\mathcal{A}^\omega$ has minimal closure (say $\mathfrak{B}$, where $\mathfrak{B} = (X, \{B_i\})$), then the covering configuration is isomorphic to the extended Q-bipartite double of $\mathfrak{B}$, when it exists, if the rank of $\mathfrak{B}$ is odd ([18, 3.1]). Existence requires $\mathfrak{B}$ to be cometric with an additional condition on the Krein parameters. For even rank, the covering configuration has a fusion (merging just two classes) that is isomorphic to the extended Q-bipartite double, provided that there is exactly one class of $\mathfrak{A}$ on which ω is constant. Note that a minimal closure of a weight with values in U_2 has even rank only when the weight is constant on an odd number of classes of $\mathfrak{A}$. The isomorphism is $M \otimes N \mapsto N \otimes M$ on the $C_{i\sigma}$ of the cover configuration.

4.2.1 Necessary conditions for a covering configuration

In the case of commutative CCs we extend [16, Prop. 5.4] in a natural way, as follows.

Let $\mathfrak{C} = (X, \{R_i\})$ be a commutative CC of rank tr such that the first t intersection matrices have the form $M_j = I_r \otimes Z_{\zeta^j}$, for $0 \le j < t$, and let $U = \langle\zeta\rangle$ the group of roots of unity of order t. Index the relations according to the r blocks of size t, so that

$$C_{i,\zeta^k} = R_{it+k}$$

and suppose that for any i, j, k and ν:

$$p_{i\sigma,j\tau}^{k\sigma\tau\nu} = p_{i1,j1}^{k\nu}$$

for all σ and τ in U. We intend to show that under these conditions, $\mathfrak{C}$ must arise as the covering configuration of a regular weight on a quotient of $\mathfrak{C}$.

Lemma 4.4. *If $j < t$ and $p_{ij}^k \neq 0$, then $k = i + j \pmod t$; in particular, i and k lie in the same block of M_j.*

Proof. This follows from $M_j = I \otimes Z_{\zeta^j}$. □

Observe that $E := \cup_{j=0}^{t-1} R_j$ is a *parabolic* in the sense of [10]. Indeed, $M_0 = I_{rt}$ implies that R_0 is the identity relation of $\mathfrak{C}$. Further, E is symmetric, since $(x, y) \in R_i$ for $i < t$ implies that $p_{i^*i}^0 \neq 0$, so i^* is in the same block of M_i as 0. That is, $(y, x) \in E$. Given $(x, y) \in R_i$ and $(y, z) \in R_j$ with $0 \leq i, j < t$, we see that $(x, z) \in R_k$ for some $k < t$, because k must lie in the same block of M_j as i, since all non-diagonal blocks are zero. Hence, E is a transitive relation.

As a parabolic, E induces an equivalence relation on the indices: If there exist x, x', y, $y' \in X$ such that $(x, x') \in E$, $(y, y') \in E$, $(x, y) \in R_i$ and $(x', y') \in R_j$, then $i \sim j$. Write $[i]$ for the equivalence class of i. In addition, the parabolic affords a quotient (homogeneous) configuration $\mathfrak{A} := (\overline{X}, \{\overline{R}_{[i]}\})$ with an associated partition of the vertex set X into *fibres* of size t. The fibre containing x is

$$[x] = \{y \mid (x, y) \in E\}.$$

We will henceforth suppress the bracket notation for fibres, writing $x = \{x_1, x_2, \ldots x_t\}$.

For $j \in [0]$, Lemma 4.4 implies that $p_{kj}^k = 0$ for $j \neq 0$. But then R_k restricted to $x \times y$ has valency at most 1. We conclude that the number of relations occurring between any two fibres is t. We have: For $k \in \mathcal{I}$ and $x \in X$,

$$|[k]| = |x| = t.$$

Denoting the graph of R_j by Γ_j, we have proved the following:

Lemma 4.5. *For all $j \notin [0]$, Γ_j is a t-fold cover of $\Gamma_{[j]}$.*

Corollary 4.6. *The natural partition of $\mathcal{I}$ according to blocks of M_j, for $0 \leq j < t$ is the same as that determined by the equivalence classes of the parabolic. That is,*

$$[mt] = \{mt, mt + 1, \ldots, mt + t - 1\}.$$

Proof. Suppose $j \in [i]$ so that there exist $x_1, x_2, y_1, y_2 \in X$ with $(x_1, y_1) \in R_i$ and $(x_2, y_2) \in R_j$. Then, by the discussion above, $(x_1, y_3) \in R_j$ for some $y_3 \in y$ and therefore $p_{ik}^j \neq 0$ for some $k < t$.

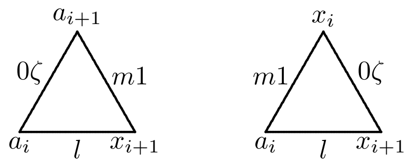
But then $j = i + k \pmod t$ by Lemma 4.4. □

Recall that $C_{0,\zeta^k} = R_k$ for $k < t$, $C_{0,\sigma}$ has intersection matrix $I_r \otimes Z_\sigma$, and $C_{m,1} = R_{mt}$ for $0 \leq m < r$. Fix a fibre a (from here on), and order it so that $(a_i, a_{i+1}) \in C_{0\zeta}$, for each i, with addition modulo t. This ensures that the perfect matching induced on a corresponds to the permutation $(1, 2, \dots, t)$ on indices, which in turn corresponds to the permutation of U induced by multiplication by ζ.

For each $x \in \overline{X}$, $(a, x) \in \overline{R}_{[mt]}$ for some m. Order x so that $(a_j, x_j) \in C_{m,1}$. In what follows, we mix the notations regarding indexation of the relations of $\mathfrak{C}$. Where two indices are given, we refer to $C_{i,\sigma}$ as above; where one index is given we refer to the original numbering of the relations.

Lemma 4.7. *With notation as above,* $(x_i, x_{i+1}) \in C_{0,\zeta}$ *for all* $x \in \overline{X}$.

Proof. For some σ, $(x_i, x_{i+1}) \in C_{0,\sigma}$; $(a_i, x_{i+1}) \in R_l$ for some l, and $(a_i, x_i) \in R_{m1}$ for some m. Note that $l \in [m]$. Since a_i, a_{i+1}, and x_{i+1} form a triangle of type $(0\zeta, m1, l)$,



we see that $p^l_{0\zeta,m1} \neq 0$. Since $\mathfrak{C}$ is commutative, $R_l = C_{m\zeta}$ by Lemma 4.4. Now observe that a_i, x_i, and x_{i+1} form a triangle of type $(m1, 0\sigma, m\zeta)$, and therefore $\sigma = \zeta$. □

Next, following [16] we show that all matchings are cyclic.

Lemma 4.8. *With notation as above, all matchings between fibres of* $\mathfrak{C}$ *are cyclic.*

Proof. Suppose that $(x_i, y_j) \in R_k$ and $(x_{i+1}, y_l) \in R_k$. We must show that $l = j + 1$. The triangle (x_i, x_{i+1}, y_j) has type $(1, m, k)$ for some m, indicating that $p^k_{1m} \neq 0$. As in the previous lemma, this implies that $k = m + 1$. On the other hand, the triangle (x_{i+1}, y_{l-1}, y_l) has type $(b, 1, k)$ for some b, hence $k = b + 1$. But then $m = b$, and by Lemma 4.5, $y_{l-1} = y_j$ as desired. □

Corollary 4.9. *For all* $x \in X$, $(x_i, x_{i+k}) \in R_k$, *thus* R_k *induces on each fibre the perfect matching corresponding to the* k^{th} *power of the cycle* $(1, , \dots, t)$.

Proof. The result follows by Lemma 4.7 and induction (on k) applied to the triangles (x_{i-k}, x_i, x_{i+1}) . □

Lemma 4.10. *For* $x \in X$, $(a_i, x_{i+k}) \in R_{mt+k}$ *for* $0 \leq k < t$.

Proof. The case $k = 0$ holds by choice of ordering of x. Induction applied to the triangles $(a_i, x_{i+k-1}, x_{i+k})$ gives the desired result. □

We now define a weight on $\mathfrak{A}$ by means of C_{i1}. Let $x, y \in \overline{X}$ and suppose $(x, y) \in \overline{R}_{[j]}$. Then $C_{j,1}$ provides a cyclic matching between x and y corresponding to, say, $\alpha \in U$. Set $\omega(x, y) := \alpha$. Observe that $\omega(a, x) = 1$ for all x.

The next lemma shows how to determine the weight of an edge in $\overline{\Gamma}_{[i]}$ from any edge in Γ_i.

Lemma 4.11. *If $(x_i, y_j) \in C_{k\sigma}$, then $\omega(x, y) = \bar{\sigma}\zeta^{j-i}$.*

Proof. Consider $(x_i, y_j) \in C_{k,\sigma}$. Let l be such that $(x_i, y_l) \in C_{k1}$ and note that the triangle (x_i, y_l, y_j) has type $(k1, 0\zeta^{j-l}, k\sigma)$. By Proposition 4.6, $\sigma = \zeta^{j-l}$. This implies that $(x_i, y_{l+m}) \in C_{k,\zeta^m}$. We conclude that the matching between x and y in $C_{k,\sigma}$ is $\alpha\sigma$, where $\alpha = \omega(x, y)$. □

We now prove the second main result which is the extension of [16, Prop. 5.4].

Theorem 4.12. *Let $\mathfrak{C} = (X, \{R_i\})$ be a commutative CC of rank rt with the first t intersection matrices given by*

$$M_j = I_r \otimes Z_{\zeta^j} \quad 0 \le j < t,$$

where $U = U_t = \langle\zeta\rangle$ is the group of roots of unity of order t. Label the relations according to the blocking of M_j:

$$C_{i,\zeta^k} := R_{it+k} \quad 0 \le i < r, 0 \le k < t$$

and suppose that the CC parameters satisfy, for any i, j, k and ν:

$$p_{i\sigma,j\tau}^{k\sigma\tau\nu} = p_{i1,j1}^{k\nu}$$

for all σ and τ in U. Then $\mathfrak{C}$ arises as the covering configuration (in the sense Theorem 3.1) from a regular weight ω on the quotient scheme $\mathfrak{A} = \mathfrak{C}/E$.

Proof. From the discussion and lemmas above, what remains to be shown is that ω is regular on the quotient configuration $\overline{\mathfrak{C}} = (\overline{X}, \{\overline{R}_{[i]}\})$. Let $(x, z) \in \overline{R}_{[k]}$. We consider all y such that (x, y, z) has type (i, j, k) and weight ν. Let l be such that $(x_1, z_l) \in C_{k\nu}$. If $(x, y) \in \overline{R}_{[i]}$ and $(y, z) \in \overline{R}_{[j]}$, then $(x_1, y_m) \in C_{i1}$ for some m, and this determines (exactly one) τ with $(y_m, z_l) \in C_{j\tau}$. By Lemma 4.11,

$$\begin{aligned}\delta\omega(x, y, z) &= \omega(x, y)\omega(y, z)\overline{\omega(x, z)} \\ &= \zeta^{m-1}\bar{\tau}\zeta^{l-m}\nu\zeta^{1-l} \\ &= \bar{\tau}\nu\end{aligned}$$

from which we see that triangles of weight ν occur exactly when $\tau = 1$. These triangles are counted by the parameter $p_{i1,j1}^{k\nu}$ which is independent of the choice of x_1 and z_l. □

Note that in the proof above we are counting distinct y, and that for each y there is exactly one y_m as indicated. Thus we may use C_{i1} without loss of generality, since $(x_1, y_m) \in C_{i\sigma}$ would yield the same result. In fact triangles of type $(i\sigma, j\tau, k\nu)$ will have weight ν exactly when $\sigma\tau = 1$, which is expected as in that case $p_{i\sigma,j\tau}^{k\nu} = p_{i1,j1}^{k\nu}$

5 Examples

5.1 A rank 12 scheme on 18 points

The covering configuration of Example 2.6 has rank 12 (= 4 · 3) on 18 (= 6 · 3) points. It is isomorphic to as18[88] on Hanaki and Izumi's list ([8]).

5.2 A family of CCs from regular weights on $H(n,2)$ with values in U_4

This construction is due to Ada Chan (personal communication). We define a regular weight on the Hamming Scheme $H(n,2)$ with values in U_4 with generator $\mathbf{i}$. Let $\mathbf{t}$ be an indeterminate, and K the 2 by 2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Form $(I+\mathbf{t}K)^{\otimes n}$, a polynomial in $\mathbf{t}$ with coefficients in the ring of matrices $M_{2,2}(\mathbb{R})^{\otimes n} \simeq M_{2^n,2^n}(\mathbb{R})$. Now let A_k^ω be the coefficient of $\mathbf{t}^k$, scaled by a factor of $\mathbf{i}^k \in U_4$. We claim this is a regular weight on the Hamming scheme. Indeed, replacing $\mathbf{i}$ with 1 and K with $J-I$ in this process yields the adjacency matrices of the Hamming scheme, with the standard P-polynomial ordering. Noting that $K^2 = -I$ it is straight-forward to see that Span(A_k^ω) is coherent. For regularity, we note that p_{ij}^k is nonzero only when $i+j+k$ is even, and this implies $\beta_{ij}^k(\pm\mathbf{i}) = 0$ for all i,j,k. Proposition 1 of [21] applies, and we conclude that ω is regular.

The covering configuration induced by this weight is a rank $4(n+1)$ CC on 2^{n+2} vertices. There is a fusion to regular 4-graph, which is easily seen: replace $\mathbf{t}$ by $\mathbf{i}$, setting

$$\widetilde{\omega} := (I+\mathbf{i}K)^{\otimes n},$$

then verify directly that $\widetilde{\omega}^2 = 2^n\widetilde{\omega}$ thus $\widetilde{\omega}$ is the matrix of a regular 4-graph. The covering configuration of $\widetilde{\omega}$ has rank 8 and is symmetric, but not necessarily distance regular.

For $n=2$, the weight is given by:

$$A_1^\omega = \begin{bmatrix} 0 & i & i & 0 \\ -i & 0 & 0 & i \\ -i & 0 & 0 & i \\ 0 & -i & -i & 0 \end{bmatrix} \quad \text{and} \quad A_2^\omega = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The rank 12 covering configuration has color matrix $(\sum iA_i)$ below.

$$\begin{bmatrix}
0 & 1 & 2 & 3 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 8 & 9 & 10 & 11 \\
3 & 0 & 1 & 2 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 11 & 8 & 9 & 10 \\
2 & 3 & 0 & 1 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 10 & 11 & 8 & 9 \\
1 & 2 & 3 & 0 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 8 \\
5 & 6 & 7 & 4 & 0 & 1 & 2 & 3 & 10 & 11 & 8 & 9 & 7 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 & 3 & 0 & 1 & 2 & 9 & 10 & 11 & 8 & 6 & 7 & 4 & 5 \\
7 & 4 & 5 & 6 & 2 & 3 & 0 & 1 & 8 & 9 & 10 & 11 & 5 & 6 & 7 & 4 \\
6 & 7 & 4 & 5 & 1 & 2 & 3 & 0 & 11 & 8 & 9 & 10 & 4 & 5 & 6 & 7 \\
5 & 6 & 7 & 4 & 10 & 11 & 8 & 9 & 0 & 1 & 2 & 3 & 7 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 & 9 & 10 & 11 & 8 & 3 & 0 & 1 & 2 & 6 & 7 & 4 & 5 \\
7 & 4 & 5 & 6 & 8 & 9 & 10 & 11 & 2 & 3 & 0 & 1 & 5 & 6 & 7 & 4 \\
6 & 7 & 4 & 5 & 11 & 8 & 9 & 10 & 1 & 2 & 3 & 0 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 0 & 1 & 2 & 3 \\
11 & 8 & 9 & 10 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 3 & 0 & 1 & 2 \\
10 & 11 & 8 & 9 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 2 & 3 & 0 & 1 \\
9 & 10 & 11 & 8 & 6 & 7 & 4 & 5 & 6 & 7 & 4 & 5 & 1 & 2 & 3 & 0
\end{bmatrix}$$

The regular 4-graph $\widetilde{\omega} := I + A_1^\omega + A_2^\omega$ satisfies $\widetilde{\omega}^2 = 4I$. The covering configuration of $\widetilde{\omega}$ has rank 8 and may also be obtained through fusion of the rank 12 above.

In summary, this construction gives regular weights with values in U_4 on the Hamming Schemes $\mathrm{H}(n, 2)$. These have rank $n + 1$ on 2^n vertices. The covering configurations thus have rank $4(n+1)$ on 2^{n+2} vertices. These weights fuse to regular 4-graphs always, and the covering configurations of those have rank 8. In examples constructed to date, the covering configurations are not metric, nor are their symmetrizations, and they are not cometric.

5.3 CCs afforded by groups

A CC may have relations determined by the orbitals of a group G acting on a set X, in which the centralizer algebra of the natural permutation representation is the coherent algebra $\mathcal{A}$. In this case, a regular weight may exist such that $\mathcal{A}^\omega$ is the centralizer algebra of a monomial representation of G, induced from a linear representation of a point stabilizer ([14]).

For example, the rank 3 scheme containing the Petersen graph is afforded by the action of A_5 on 2-sets from $\{1, 2, 3, 4, 5\}$. The stabilizer of $\{1, 2\}$ is a group $H \simeq S_3$, containing $A := \langle 3, 4, 5\rangle$ as a subgroup of index 2. This index determines that the monomial representation will afford a weight with values in U_2. Defining the linear representation

$$\phi : H \to C_2 \quad \text{by} \quad \phi(g) = \begin{cases} 1 & g \in A, \\ -1 & g \notin A, \end{cases}$$

the induced representation $M := \phi\big|_H^G$ is a monomial representation of G. The $M(g)$ for $g \in G$ are signed permutation matrices. The centralizer algebra of M, $\mathcal{A}^\omega$, defines a regular weight on the Petersen graph.

This construction can be done in general when the point stabilizer H has a normal subgroup A of index t, such that $H/A \simeq C_t$. The monomial representation induced may or may not afford a nontrivial regular weight on the underlying CC.

In this example, the covering configuration $\mathfrak{C}$ is a rank 6 scheme on 20 points, in fact the unique (antipodal, non-bipartite) distance-regular graph DRG$\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$, that is the dodecahedron graph. (It is not the bipartite double of the Petersen graph, which is DRG$\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}$.)

We obtain a permutation representation from M, via

$$M(g) \mapsto M^+(g) \otimes Z_1 + M^-(g) \otimes Z_2$$

where M^+, M^-, Z_1 and Z_2 are defined as in Section 2. It is natural to ask whether $\mathcal{C}$ is the centralizer algebra of this permutation representation. In fact, $\mathcal{C}$ is properly contained in this centralizer algebra. It affords a CC with valencies 1, 1, 3, 3, 3, 3, 3, 3 which has a fusion to $\mathfrak{C}$. The group affording $\mathfrak{C}$ is $A_5 \times C_2$, an extension of our group G by the cyclic C_2, the latter generated by the even permutation interchanging each x_1 and x_2. This is of course the symmetry group of the dodecahedron and is not isomorphic to S_5.

References

References

[1] E. Bannai and T. Ito, *Algebraic combinatorics. I: Association Schemes*, The Benjamin/Cummings Publishing Co. Inc., Menlo Park, CA, 1984.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance Regular Graphs*, Springer-Verlag, 1989.

[3] P. J. Cameron, J. M. Goethals and J. Seidel, Strongly Regular Graphs Having Strongly Regular Subconstituents, *J. Algebra* **55** (1978), 257–280.

[4] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, volume 22 of *London Math. Soc. Student Texts*, Cambridge University Press, Cambridge and New York, 1991.

[5] C. D. Godsil, *Algebraic combinatorics*, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993.

[6] C. D. Godsil and A. D. Hensel, Distance regular covers of the complete graph, *J. Combin. Theory Ser. B* **56** (1992), 205–238.

[7] W. H. Haemers and D. G. Higman, Strongly regular graphs with strongly regular decomposition, *Linear Algebra Appl.* **114** (1989), 379–398.

[8] A. Hanaki and I. Miyamoto, Classification of association schemes of small order, `http://math.shinshu-u.ac.jp/~hanaki/as/`, accessed August, 2015.

[9] D. G. Higman, A note on regular 3-graphs, unpublished draft.

[10] D. G. Higman, The parabolics of a semi-coherent configuration, unpublished draft.

[11] D. G. Higman, Coherent configurations I: ordinary representation theory, *Geom. Dedicata* **4** (1975), 1–32.

[12] D. G. Higman, Coherent algebras, *Linear Algebra Appl.* **93** (1987), 209–239.

[13] D. G. Higman, Strongly regular designs and coherent configurations of type [3,2;3], *European J. Combin.* **9** (1988), 411–422.

[14] D. G. Higman, Weights and t-graphs, *Bull. Soc. Math. Belg. Sér. A* **42** (1990), 501–521.

[15] D. G. Higman, Rank 5 association schemes and triality, *Linear Algebra Appl.* **226-228** (1995), 197–222.

[16] D. Kalmanovich, On D. G. Higman's note on regular 3-graphs, *Ars Math. Contemp.* **6** (2013), 99–115.

[17] M. Klin, A. Munemasa, M. Muzychuk and P.-H. Zieschang, Directed strongly regular graphs obtained from coherent algebras, *Linear Algebra Appl.* **377** (2004), 83–109.

[18] W. J. Martin, M. Muzychuk and J. Williford, Imprimitive cometric association schemes: Constructions and analysis, *J. Algebraic Combin.* **25** (2007), 399–415.

[19] W. J. Martin and H. Tanaka, Commutative association schemes, *Eur. J. Combin.* **30** (2009), 1497–1524.

[20] A. D. Sankey, Regular weights of full rank on strongly regular graphs, *Israel J. Math.* **95** (1996), 1–23.

[21] A. D. Sankey, Weighted association schemes, fusions, and minimal coherent closures, *J. Algebraic Combin.* (2014), 1–31, doi:10.1007/s10801-014-0553-2.

[22] J. J. Seidel, A survey of two-graphs, in: *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo I*, Accad. Naz. Lincei, Rome, pp. 481–511. Atti dei Convegni Lincei, No. 17, 1976.

[23] J. J. Seidel, More about two-graphs, in: *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity (Prachatice, 1990)*, North-Holland, Amsterdam, volume 51 of *Ann. Discrete Math.*, pp. 297–308, 1992.

[24] J. J. Seidel and D. E. Taylor, Two-graphs, a second survey, in: *Algebraic methods in graph theory, Vol. I, II (Szeged, 1978)*, North-Holland, Amsterdam, volume 25 of *Colloq. Math. Soc. János Bolyai*, pp. 689–711, 1981.

[25] D. E. Taylor, Regular 2-graphs, *Proc. London Math. Soc. (3)* **35** (1977), 257–274.

[26] D. E. Taylor and R. Levingston, Distance-regular graphs, in: *Combinatorial Mathematics (Proc. Internat. Conf. Combinatorial Theory, Australian Nat. Univ. Canberra, 1977)*, Springer, Berlin, volume 686 of *Lecture Notes in Math.*, pp. 313–323, 1978.

[27] E. R. van Dam, J. H. Koolen and H. Tanaka, Distance-regular graphs, `arXiv:1410.6294[mathCO]`, 2016.

[28] B. J. Weisfeiler and A. A. Lehman, Reduction of a graph to a canonical form and an algebra which appears in the process, *NTI* **9** (1968), 12–16.