ON STRONGLY REGULAR DESIGNS ADMITTING FUSION TO STRONGLY REGULAR DECOMPOSITION

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Abstract. A strongly regular decomposition of a strongly regular graph is a partition of the vertex set into two parts on which the induced subgraphs are strongly regular, or cliques or cocliques. Strongly regular designs as defined by D.G. Higman in [12] are coherent configurations of rank 10 with two fibers in which the homogeneous configuration on each fiber is a strongly regular graph. Haemers and Higman in [9] proved the equivalence between strongly regular decompositions, excluding special cases, and strongly regular designs with certain parameter conditions. Here we obtain this result by examining the strongly regular designs that admit a fusion to a strongly regular graph on the full vertex set. We derive equivalent conditions to Theorem 2.8 of [9] by elementary methods. Incorporating recent work of Hanaki in [10] and Kin and Reichard in [16] and [15], a table of feasible parameter sets for this class of strongly regular designs is presented along with a discussion of known constructions. In two cases, non-existence is observed due to nonexistence of the strongly regular graph obtained through fusion. One of these is also ruled out by Hobart's generalised Krein conditions, applied to strongly regular designs ([13]). As strongly regular decompositions of the complete graph have sparked interest with recent papers ([14], [18], [21]) we observe that in our situation this occurs only when the constituent graphs are also complete and the design is trivial.

1. Introduction

Coherent configurations with one fiber are (symmetric or not; commutative or not) association schemes. To advance the classification of cc's of small type, Higman considered nontrivial cc's with two fibers, and maximum rank 3. They are represented by type matrices

$$\begin{bmatrix} 2 & 2 \\ & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 3 \\ & 3 \end{bmatrix}.$$

The first two are objects well studied in design theory, namely symmetric designs and quasi-symmetric designs. The third is a subclass of the $1\frac{1}{2}$-designs (see [19]), also called partial geometric designs, and is the subject of [12]. Conditions developed there permit the calculation of feasible parameter sets. Hanaki in unpublished work ([10]) has corrected typos in that work, added some proofs that were omitted, and computed the structure constants of an srd explicitly, in terms of a core set of 14 parameters.

Key words and phrases. strongly regular design, strongly regular decomposition, coherent configuration, strongly regular graph.

Klin and Reichard in [16] embarked on the enumeration of "small" srd's, making use of strongly regular graphs with known constructive enumeration. Results are tabulated in [15] for up to 35 vertices in each of the two fibers, with enumeration settled for many cases on the list.

Strongly regular graphs with strongly regular decomposition were investigated (in fact, that is the title of the paper) by Haemers and Higman in [9]. Through eigenvalue techniques such as interlacing they developed strong parameter conditions leading to feasible parameter sets with $n_1, n_2 < 300$. Families of examples are described in that work, involving quasi-symmetric 3-designs, the symplectic graphs, and hemisystems of which only one was known at the time. Leaving aside cases in which the srg Γ_0 is imprimitive or either of its constituent graphs Γ_1 and Γ_2 is a clique or coqlique, a strongly regular decomposition is called *exceptional* when its eigenvalues are distinct from those of Γ_1 and Γ_2. This implies, in particular, that Γ_0 is the graph of a regular symmetric conference matrix. In all other cases, Γ_0, Γ_1, and Γ_2 determine a nontrivial srd. It is this class of srd's that we investigate here. These srd's, considered as rank 10 cc's, have Γ_0 as a (rank 3) fusion scheme.

Beginning with the intersection numbers (from Hanaki) for an srd, it is straight-forward to determine the conditions under which a fusion produces an srg which has both Γ_1 and Γ_2 as induced subgraphs. This is done in Section 3. In Section 4 we show that our conditions mirror those of [9], Theorem 2.8. We briefly discuss the fusion to a rank 5 association scheme that occurs if and only if the srd is symmetric, in Section 5. These schemes are exploited in some of the constructions found in [16] and are shown to be cometric and Q-antipodal association schemes, in work of van Dam, Martin and Muzychuk ([22]). In Section 6 we consider the table of feasible parameters and discuss existence, nonexistence, and constructions. In the final section, we treat the case in which Γ_0 is a complete graph.

1.1. **A first example.** A *strongly regular graph* with parameters (n, k, λ, μ) is a regular graph on n vertices of degree k, with the property that two distinct vertices have λ or μ common neighbours depending on whether they are adjacent or not.

For a simple example, consider srg's (28, 12, 6, 4) and (35, 16, 6, 6). The first may be taken to be the triangular graph $T(8)$, with unordered pairs from the set $\{1, 2, 3, \ldots, 8\}$ as vertices and adjacency given by nontrivial intersection. (This is also the distance 1 graph in the Johnson scheme J(8,2).) The second graph may be realised by the bisections — partitions into two parts of size 4 — of the symbols 1 through 8, with adjacency corresponding to set intersections of sizes 1 and 3. For example, (simplifying the notation) the bisection {1234, 5678} is adjacent to {1567, 2348} and not to {1256, 3478}. The bisections are in one-to-one correspondence with the 3-sets from 2 through 8 via the bijection

$$\{1abc, defg\} \longleftrightarrow \{abc\}.$$

On this set we have a Johnson scheme J(7,3), in which the graph above is obtained as the fusion of the distance 1 and 3 graphs. The reader unfamiliar with the particulars of association schemes or coherent configurations is referred to the references [3], [5], [7], [11], and [17]. However, the essential definitions are given in Section 2 so as to make the present work more or less self-contained.

We now have two srg's with disjoint vertex sets and we wish to define an incidence structure with vertices of Γ_1 serving as "points" and those of Γ_2 as "blocks". This is obtained via the subset relation: we call a 2-set $\{ab\}$ *incident* with a bisection provided it is a subset of one of the parts. For instance, {58} is incident with {1234, 5678} but not {1235, 4678}. It is now easily confirmed that there are:

- 12 points in each block and 15 blocks containing each point;
- two possible block intersection sizes, depending on whether the blocks are adjacent in Γ_2: 6 and 4;
- two possible values for the number of common blocks to two points, and this depends on whether the points are adjacent in Γ_1: 5 and 7;
- two possible values for the number of points adjacent to a point x and incident with a block y, depending on whether x is incident with y: 4 and 6;
- two possible values for the number of blocks containing a point x and adjacent to a block y, depending on whether x is incident with y: 8 and 6.

The importance of the above is that these values are some of the intersection numbers (the p_{ij}^k parameters) of a coherent configuration with the point and block sets as fibers; the remaining parameters are derived from these. Thus we have a cc of type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$.

It should be noted that this parameter set is also a confirmed "group case" in [12], a configuration afforded by $L_4(2) \simeq A_8$.

2. Preliminaries

Defn 2.1. *A strongly regular design is a finite incidence structure consisting of a set X_1 of points, a set X_2 of blocks, and an incidence relation $F \subseteq X_1 \times X_2$, such that the following are nonnegative integer constants:*

- S_1 := *number of points incident with (in) each block;*
- S_2 := *number of blocks incident with (containing) each point;*
- a_1, b_1 := *the two distinct block intersection sizes;*
- a_2, b_2 := *the two distinct point join sizes, that is the number of blocks containing two given points;*
- N_1 (P_1) := *number of points adjacent to a point x and incident with a block y, provided x is (is not) incident with y;*
- N_2 (P_2) := *number of blocks containing a point x and adjacent to a block y, given x is (is not) incident with y.*

Adjacency in the block graph means the blocks intersect in a_1 points; two points likewise are adjacent in the point graph if and only if they lie in a_2 common blocks. It follows immediately from the definition that both the point graph and the block graph are strongly regular.

Let C be the 0/1 incidence matrix with rows indexed by the $n_1 := |X_1|$ points and columns by the $n_2 := |X_2|$ blocks. Then, letting J be the all ones matrix of the appropriate dimensions, we have ([12]):

(i) C has row sum S_2 and column sum S_1;

(ii) $CC^T = (S_2 - b_2)I + (a_2 - b_2)A_1 + b_2 J$;
(iii) $C^T C = (S_1 - b_1)I + (a_1 - b_1)A_2 + b_1 J$;
(iv) $CA_2 = (N_2 - P_2)C + P_2 J$;
(v) $A_1 C = (N_1 - P_1)C + P_1 J$.

Here A_1 and A_2 are the adjacency matrices of the point graph, Γ_1, and the block graph, Γ_2, respectively. The srg parameters are determined from their eigenvalues

$$k_1 = \frac{S_2(S_1 - 1) - b_2(n_1 - 1)}{a_2 - b_2}, \qquad \{r_1, s_1\} = \left\{ N_1 - P_1, -\frac{S_2 - b_2}{a_2 - b_2} \right\} \quad \text{and}$$

$$k_2 = \frac{S_1(S_2 - 1) - b_1(n_2 - 1)}{a_1 - b_1}, \qquad \{r_2, s_2\} = \left\{ N_2 - P_2, -\frac{S_1 - b_1}{a_1 - b_1} \right\}$$

in the usual way: $\mu_i = k_i + r_i s_i$, $\lambda_i = \mu_i + r_i + s_i$. Set $l_i = n_i - k_i - 1$. Multiplicities for the eigenvalues r and s of an srg are computed as

$$f := \frac{(n-1)(-s) - k}{r - s}, \qquad g := n - f - 1$$

where s is the negative eigenvalue, but for srd's we will not assume $s_i < 0$ as there are examples of both $N_1 > P_1$ and $N_1 < P_1$.

An srg is *imprimitive* if the graph or its complement is disconnected. Imprimitivity of a strongly regular design occurs when either there are repeated blocks, so that $S_1 = b_1$ or there are "repeated points" meaning $S_2 = b_2$. In the first case, we get $s_2 = -1$, and in the second $s_1 = -1$, which correspond to imprimitivity of the block graph or the point graph respectively.

We now have the pieces in place to confirm that an srd is a *coherent configuration*, as defined below.

Defn 2.2. *Let $\{A_i\}_{0 \le i < r}$ be a set of $0/1$-matrices with rows and columns indexed by a finite set X. Let $\mathcal{I} := \{0, 1, \ldots, r-1\}$. The linear span $\mathcal{A} := \langle A_i \rangle_{\mathbb{C}}$ is a coherent algebra if:*

(i) $\sum_{i \in \mathcal{I}} A_i = J$, *where J is the all-ones matrix,*
(ii) $\sum_{i \in \mathcal{L}} A_i = I$, *for some subset $\mathcal{L} \subset \mathcal{I}$,*
(iii) for each i there exists $i^* \in \mathcal{I}$ such that $A_i^T = A_{i^*}$,
(iv) $A_i A_j = \sum p_{ij}^k A_k$, $p_{ij}^k \in \mathbb{Z}^+$.

The set $\mathcal{L}$ consists of those matrices with 1's on the diagonal corresponding to some subset of X, and 0's elsewhere. This induces a partition of the vertex set into *fibers*. A consequence of the definition is that the A_i's may be blocked according to these fibers such that no A_i is nonzero in more than two blocks, and not more than one if it is not symmetric. The *type* of a cc is a matrix indicating how many indices i appear in each block. A strongly regular design is a cc of type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$, for example, which indicates two fibers X_1 and X_2, with 3 relations on each representing the strongly regular point and block graphs, and 2 relations – incidence and non-incidence – on point-block pairs plus their transposes on block-point pairs. This cc therefore has *rank* $r = 10$.

A coherent algebra is *homogeneous* if $|\mathcal{L}| = 1$; *symmetric* if $i^* = i$ for all i, and *commutative*, clearly, if $p^k_{ij} = p^k_{ji}$ for all i, j, k. The homogeneous coherent algebras are (possibly non-symmetric) *association schemes*. Commutative schemes which have the *metric* property are synonymous with *distance-regular graphs*; those of diameter 2 are the strongly regular graphs.

In the association scheme literature, a rank r scheme is often referred to as an $(r-1)$-class scheme: 'rank' counts the trivial relation, while the number of 'classes' does not. The indexing set $\mathcal{I} = \{1, 2, \ldots, r\}$ is sometimes used in place of 0 through $r-1$.

Every algebra of n by n matrices over $\mathbb{C}$ that is closed under transpose and entry-wise multiplication, and contains both I and J is a coherent algebra, and as such it has a basis of $0/1$-matrices satisfying (i)–(iv). Each A_i in a coherent algebra is the adjacency matrix of a digraph Γ_i with vertex set X, which is simple for $i \notin \mathcal{L}$ and undirected when $i^* = i$. Viewing these graphs as relations on X, we define a *coherent configuration* (cc) to be a set of binary relations on X, indexed by $\mathcal{I}$, with analogous properties to (i)–(iv) above. Denote it $\mathfrak{A} := (X, \{R_i\}_{i \in \mathcal{I}})$.

The *intersection matrices* $\mathbf{M}_j$ of a cc are the $r \times r$ matrices $\mathbf{M}_j := \left(p^k_{ij}\right), i, k \in \mathcal{I}$ and the map

$$\gamma : \mathbf{A}_j \mapsto \mathbf{M}_j$$

is the right regular representation of $\mathcal{A}$.

We treat coherent algebras and cc's as equivalent structures and move freely between the notations of matrices, relations, and graphs. As $\{A_i\}$ forms the *standard basis* of $\mathcal{A}$, we refer to $\{R_i\}$ and $\{\Gamma_i\}$ as the *basic relations* and *basic graphs* of $\mathfrak{A}$ respectively. This facilitates interpretations such as noting that the structure constant p^k_{ij} counts the number of i-j paths, meaning an edge in Γ_i followed by an edge in Γ_j, from a vertex x to a vertex z, given that $(x, z) \in R_k$. This number depends on k but not on the choice of (x, z) in R_k.

A *fusion* is a merging of relations in a cc according to a partition of $\mathcal{I}$. A fusion will be deemed *coherent* if the resulting configuration is coherent. A coherent *fission* or *refinement* is a partition of each basic relation such that the resulting set of relations forms a cc.

Returning to strongly regular designs, enumerate the relations such that 1, 2, 3 are the identity, adjacency, and non-adjacency for Γ_1; 4, 5, 6 likewise for Γ_2; 7 and 8 are incidence and non-incidence on point-block pairs; $9 = 7^*$, and $10 = 8^*$.

Dependencies are such that the six parameters $n_1, n_2, S_1, a_1, b_1, a_2$ determine the remaining values for an srd. There are 20 parameter conditions given in [12] of which the first 15 are discussed by Hanaki in [10], with corrections and proofs provided. This is also the source of the 10 by 10 intersection matrices $M_j = \left(p^k_{ij}\right)_{i,k}$ below. (There are corrections made here to $p^5_{10,8}$, $p^6_{10,8}$, $p^2_{8,10}$, and $p^3_{8,10}$).

$$M_1 = \begin{bmatrix} I_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & I_2 \end{bmatrix} M_2 = \begin{bmatrix} F_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & F_2 \end{bmatrix} M3 = \begin{bmatrix} F_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & F_4 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & I_3 & \cdot & \cdot \\ \cdot & \cdot & I_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} M_5 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & F_5 & \cdot & \cdot \\ \cdot & \cdot & F_6 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} M_6 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & F_7 & \cdot & \cdot \\ \cdot & \cdot & F_8 & \cdot \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

$$M_7 = \begin{bmatrix} \cdot & \cdot & F_9 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & F_{10} & \cdot & \cdot \end{bmatrix} M_8 = \begin{bmatrix} \cdot & \cdot & F_{11} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & F_{12} & \cdot & \cdot \end{bmatrix} M_9 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & F_{13} \\ F_{14} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} M_{10} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & F_{15} \\ F_{16} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where the blocks F_i are defined below.

$$F_1 = \begin{bmatrix} & 1 & \\ k_1 & \lambda_1 & \mu_1 \\ & k_1-\lambda_1-1 & k_1-\mu_1 \end{bmatrix}, F_2 = \begin{bmatrix} N_1 & P_1 \\ k_1-N_1 & k_1-P_1 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} & & 1 \\ & k_1-\lambda_1-1 & k_1-\mu_1 \\ l_1 & n_1-2k_1+\lambda_1 & n_1-2k_1+\mu_1-2 \end{bmatrix}, F_4 = \begin{bmatrix} S_1-N_1-1 & S_1-P_1 \\ n_1-k_1-S_1+N_1 & n_1-k_1-S_1+P_1-1 \end{bmatrix},$$

$$F_5 = \begin{bmatrix} & 1 & \\ k_2 & \lambda_2 & \mu_2 \\ & k_2-\lambda_2-1 & k_2-\mu_2 \end{bmatrix}, F_6 = \begin{bmatrix} N_2 & P_2 \\ k_2-N_2 & k_2-P_2 \end{bmatrix},$$

$$F_7 = \begin{bmatrix} & & 1 \\ & k_2-\lambda_2-1 & k_2-\mu_2 \\ l_2 & n_2-2k_2+\lambda_2 & n_2-2k_2+\mu_2-2 \end{bmatrix}, F_8 = \begin{bmatrix} S_2-N_2-1 & S_2-P_2 \\ n_2-k_2-S_2+N_2 & n_2-k_2-S_2+P_2-1 \end{bmatrix},$$

$$F_9 = \begin{bmatrix} 1 & \\ N_1 & P_1 \\ S_1-N_1-1 & S_1-P_1 \end{bmatrix}, F_{10} = \begin{bmatrix} S_1 & a_1 & b_1 \\ & S_1-a_1 & S_1-b_1 \end{bmatrix},$$

$$F_{11} = \begin{bmatrix} & 1 \\ k_1-N_1 & k_1-P_1 \\ n_1-S_1-k_1+N_1 & n_1-S_1-k_1+P_1-1 \end{bmatrix},$$

$$F_{12} = \begin{bmatrix} & S_1-a_1 & S_1-b_1 \\ n_1-S_1 & n_1-2S_1+a_1 & n_1-2S_1+b_1 \end{bmatrix}, F_{13} = \begin{bmatrix} 1 & \\ N_2 & P_2 \\ S_2-N_2-1 & S_2-P_2 \end{bmatrix},$$

$$F_{14} = \begin{bmatrix} S_2 & a_2 & b_2 \\ & S_2-a_2 & S_2-b_2 \end{bmatrix}, F_{15} = \begin{bmatrix} & 1 \\ k_2-N_2 & k_2-P_2 \\ n_2-S_2-k_2+N_2 & n_2-S_2-k_2+P_2-1 \end{bmatrix},$$

$$F_{16} = \begin{bmatrix} & S_2-a_2 & S_2-b_2 \\ n_2-S_2 & n_2-2S_2+a_2 & n_2-2S_2+b_2 \end{bmatrix}.$$

The irreducible representations of an srd are copied here from Table 1 of [13], with

$$\alpha = \sqrt{S_1 S_2}, \quad \beta = \sqrt{(n_1-S_1)(n_2-S_2)},$$

and

$$\gamma = \sqrt{S_1 + a_1 r_2 - b_1(r_2+1)}.$$

The 2 by 2 matrix E_{ij} has a single nonzero entry in row i, column j. See also [10].

	Δ_1	Δ_2	Δ_3	Δ_4
A_1	E_{11}	E_{11}	1	0
A_2	k_1E_{11}	r_1E_{11}	s_1	0
A_3	l_1E_{11}	$-(r_1+1)E_{11}$	$-(s_1+1)$	0
A_4	E_{22}	E_{22}	0	1
A_5	k_2E_{22}	r_2E_{22}	0	s_2
A_6	l_2E_{22}	$-(r_2+1)E_{22}$	0	$-(s_2+1)$
A_7	αE_{12}	γE_{12}	0	0
A_8	βE_{12}	$-\gamma E_{12}$	0	0
A_9	αE_{21}	γE_{21}	0	0
A_{10}	βE_{21}	$-\gamma E_{21}$	0	0
z_i	1	$\frac{(n_1-1)(-s_1)-k_1}{r_1-s_1}$	n_1-1-z_2	n_2-1-z_2

2.1. **Sisters.** The complement of srd$(n_i, S_i, a_i, b_i, N_i, P_i)$ according to [12] is

$$\text{srd}(n_i, n_i - S_i, n_i - 2S_i + a_i, n_i - 2S_i + b_i, k_i - P_i, k_i - N_i),$$

where the design is obtained by interchanging incidence with non-incidence (relations 7 and 9 with relations 8 and 10 respectively). The cc is of course the same. We may similarly derive the srd parameters that result from interchanging Γ_1 with its complement:

$$(n_i, S_i, a_1, a_2, b_1, b_2, N_1, N_2, P_1, P_2) \longrightarrow (n_i, S_i, a_1, b_2, b_1, a_2, S_1 - N_1 - 1, N_2, S_1 - P_1, P_2).$$

The analogous change holds for Γ_2. Once again, this amounts to only re-ordering the relations of the cc. For that reason, we do not distinguish between these eight sister srd's in the table below, but regard them as equivalent. Note that it is not possible to require both $k_i \le n_i/2$ and $b_1 < a_1$. The former is more convenient when working with tables of srg's to produce feasible srd parameters; the latter is specified in [12]. We abbreviate the parameters of $\overline{\Gamma}_i$ using $l_i := n_i - k_i - 1$, $\overline{\lambda}_i := n_i - 2k_i + \mu_i - 2$, $\overline{\mu}_i := n_i - 2k_i + \lambda_i$, and set $\overline{S}_i := n_i - S_i$, $\overline{a}_i := n_i - 2S_i + a_i$, $\overline{b}_i := n_i - 2S_i + b_i$.

srd	n_i	k_i	λ_i	μ_i	S_i	a_i	b_i	N_i	P_i
$\overline{\text{srd}}$	n_1	k_1	λ_1	μ_1	$\overline{S}_1$	$\overline{a}_1$	$\overline{b}_1$	$k_1 - P_1$	$k_1 - N_1$
	n_2	k_2	λ_2	μ_2	$\overline{S}_2$	$\overline{a}_2$	$\overline{b}_2$	$k_2 - P_2$	$k_2 - N_2$
$\overline{\Gamma}_1$	n_1	l_1	$\overline{\lambda}_1$	$\overline{\mu}_1$	S_1	a_1	b_1	$S_1 - N_1 - 1$	$S_1 - P_1$
	n_2	k_2	λ_2	μ_2	S_2	b_2	a_2	N_2	P_2
$\overline{\Gamma}_2$	n_1	k_1	λ_1	μ_1	S_1	b_1	a_1	N_1	P_1
	n_2	l_2	$\overline{\lambda}_2$	$\overline{\mu}_2$	S_2	a_2	b_2	$S_2 - N_2 - 1$	$S_2 - P_2$
$\overline{\Gamma}_1$ and $\overline{\Gamma}_2$	n_1	l_1	$\overline{\lambda}_1$	$\overline{\mu}_1$	S_1	b_1	a_1	$S_1 - N_1 - 1$	$S_1 - P_1$
	n_2	l_2	$\overline{\lambda}_2$	$\overline{\mu}_2$	S_2	b_2	a_2	$S_2 - N_2 - 1$	$S_2 - P_2$
$\overline{\text{srd}}$ and $\overline{\Gamma}_1$	n_1	l_1	$\overline{\lambda}_1$	$\overline{\mu}_1$	$\overline{S}_1$	$\overline{a}_1$	$\overline{b}_1$	$n_1 - k_1 - 1 - S_1 + P_1$	$n_1 - k1 - S_1 + N_1$
	n_2	k_2	λ_2	μ_2	$\overline{S}_2$	$\overline{b}_2$	$\overline{a}_2$	$k_2 - P_2$	$k_2 - N_2$
$\overline{\text{srd}}$ and $\overline{\Gamma}_2$	n_1	k_1	λ_1	μ_1	$\overline{S}_1$	$\overline{b}_1$	$\overline{a}_1$	$k_1 - P_1$	$k_1 - N_1$
	n_2	l_2	$\overline{\lambda}_2$	$\overline{\mu}_2$	$\overline{S}_2$	$\overline{a}_2$	$\overline{b}_2$	$n_2 - k_2 - 1 - S_2 + P_2$	$n_2 - k_2 - S_2 + N_2$
$\overline{\text{srd}}$, $\overline{\Gamma}_1$, $\overline{\Gamma}_2$	n_1	l_1	$\overline{\lambda}_1$	$\overline{\mu}_1$	$\overline{S}_1$	$\overline{b}_1$	$\overline{a}_1$	$n_1 - k_1 - 1 - S_1 + P_1$	$n_1 - k1 - S_1 + N_1$
	n_2	l_2	$\overline{\lambda}_2$	$\overline{\mu}_2$	$\overline{S}_2$	$\overline{b}_2$	$\overline{a}_2$	$n_2 - k_2 - 1 - S_2 + P_2$	$n_2 - k_2 - S_2 + N_2$

Example 2.1. *The eight sets of srd parameters given below are equivalent to the example of Section 1.1.*

n_i	k_i	λ_i	μ_i	S_i	a_i	b_i	N_i	P_i
28	12	6	4	16	10	8	6	8
35	16	6	8	20	10	12	10	8
28	12	6	4	12	6	4	4	6
35	16	6	8	15	5	7	8	6
28	15	6	10	16	10	8	9	8
35	16	6	8	20	12	10	10	8
28	12	6	4	16	8	10	6	8
35	18	9	9	20	10	12	9	12

n_i	k_i	λ_i	μ_i	S_i	a_i	b_i	N_i	P_i
28	15	6	10	16	8	10	9	8
35	18	9	9	20	12	10	9	12
28	15	6	10	12	6	4	7	6
35	16	6	8	15	7	5	8	6
28	12	6	4	12	4	6	4	6
35	18	9	9	15	5	7	6	9
28	15	6	10	12	4	6	7	6
35	18	9	9	15	7	5	6	9

2.2. **Krein conditions.** Hobart generalised the classical Krein conditions for coherent configurations in [13] and applied them to both quasisymmetic designs and strongly regular designs, demonstrating that they are stronger than the usual Krein conditions on the association schemes of the fibers. The conditions are as follows for an srd: $XY - ZW \geq 0$, where

$$X = 1 + \frac{{r_1}^3}{{k_1}^2} - \frac{(r_1+1)^3}{(n_1-k_1-1)^2}\ , \qquad Y = 1 + \frac{{r_2}^3}{{k_2}^2} - \frac{(r_2+1)^3}{(n_2-k_2-1)^2}\ ,$$

$$Z = (S_1 + a_1 r_2 - b_1(r_2+1))^3\,, \qquad W = \left(\frac{1}{S_1 S_2} - \frac{1}{(n_1-S_1)(n_2-S_2)}\right)^2.$$

Here it is assumed that $a_i > b_i$, so we replace Γ_1 and/or Γ_2 with the complement if necessary and apply this test thereby to an appropriate sister srd.

3. Fusion to strongly regular graph

Let *S* be a strongly regular design with parameters $(n_i, S_i, a_i, b_i, N_i, P_i)$ for $i = 1, 2$. Suppose there exists a fusion of the 10 relations of *S* to a rank 3 cc, such that the resulting srg Γ_0 contains Γ_1 and Γ_2 as induced subgraphs on the two fibres. Then Γ_0 is a *strongly regular decomposition* in the sense of Haemers and Higman, and the srd determined by this decomposition is *S*.

A strongly regular decomposition is said to be *proper* if neither of the induced subgraphs is a clique or a coclique. A proper strongly regular decomposition is *exceptional* provided the eigenvalues of Γ_0 are distinct from those of Γ_1 and Γ_2. In the exceptional case, Γ_0 is the graph of a regular symmetric conference matrix. We shall be primarily interested in non-exceptional decompositions.

If the edge set of Γ_0 is merely the union of the edges in Γ_1 and Γ_2 then the two subgraphs share parameters and we are clearly in the exceptional case with Γ_0 consisting of two copies of Γ_1. Otherwise, we may assume without loss of generality that each edge in Γ_0 is either an edge in one of the induced graphs, or is an incident point-block pair. Therefore, in the notation of [12] in which the relations of the srd are numbered 1 through 10, the fusion that produces Γ_0 must be according to the partition $\{1, 4\}\{2, 5, 7, 9\}\{3, 6, 8, 10\}$. Note that symmetry in Γ_0 requires that relations 7 and 9, 8 and 10 are merged, respectively.

From the intersection numbers of a coherent configuration, the feasibility of the fusion given by a partition π of the set of relations is determined by the condition below, for each ordered pair (π_a, π_b) of parts of π:

$$\sum_{i\in\pi_a, j\in\pi_b} p_{ij}^{h_1} = \sum_{i\in\pi_a, j\in\pi_b} p_{ij}^{h_2}$$

whenever h_1 and h_2 lie in the same part of π, and a and b range over all parts of π.

For the indicated fusion to Γ_0 we have $\pi_1 = \{1, 4\}$, $\pi_2 = \{2, 5, 7, 9\}$, and $\pi_3 = \{3, 6, 8, 10\}$. Feasibility may be checked efficiently by summing rows and matrices M_j (as defined in Section 2) according to π and comparing columns within each part of π. That is, letting F be the 3 by 10 matrix with standard basis vectors as columns $[e_1|e_2|e_3|e_1|e_2|e_3|e_2|e_3|e_2|e_3]$, we compute

$$F(M_1 + M_4), \qquad F(M_2 + M_5 + M_7 + M_9), \qquad F(M_3 + M_6 + M_8 + M_{10})$$

and require that columns 1 and 4, columns 2, 5, 7, 9, and columns 3, 6, 8, 10 are identical, respectively. This direct computation yields the lemma below.

Lemma 3.1. *Let $\mathcal{S}$ be an srd as above, and let Γ_0 be the graph obtained through fusion such that two vertices are adjacent if and only if they are adjacent in either Γ_1 or Γ_2 or are incident (point-block or block-point) in the design. Then Γ_0 is strongly regular with strongly regular decomposition (Γ_1, Γ_2) if and only if:*

(i) $k_1 + S_2 = k_2 + S_1$;
(ii) $\lambda_1 + a_2 = \lambda_2 + a_1 = N_1 + N_2$;
(iii) $\mu_1 + b_2 = \mu_2 + b_1 = P_1 + P_2$.

Furthermore, setting $G := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ *the intersection matrices of* Γ_0 *are then* $D_0, D_1,$ *and* D_2 *where*

$$D_0 := F(M_1 + M_4)G = I_3,$$

$$\begin{aligned} D_1 :=& F(M_2 + M_5 + M_7 + M_9)G \\ =& \begin{bmatrix} & 1 & \\ k_1 + S_2 & \lambda_1 + a_2 & \mu_1 + b_2 \\ & k_1 + S_2 - \lambda_1 - a_2 - 1 & k_1 + S_2 - \mu_1 - b_2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} D_2 :=& F(M_3 + M_6 + M_8 + M_{10})G \\ =& \begin{bmatrix} & & 1 \\ & k_1 + S_2 - \lambda_1 - a_2 - 1 & k_1 + S_2 - \mu_1 - b_2 \\ l_1 + n_2 - S_2 & n_1 + n_2 - 2(k_1 + S_2) + \lambda_1 + a_2 & n_1 + n_2 - 2(k_1 + S_2) + \mu_1 + b_2 \end{bmatrix}. \end{aligned}$$

We infer from D_1 that the srg parameters of Γ_0 are $(n_1 + n_2, k_1 + S_2, \lambda_1 + a_2, \mu_1 + b_2)$.

Note: Only two of the eight sister srd's of Section 2.1 meet conditions (i)–(iii) of Lemma 3.1 and only one with the additional requirement that $k_i \le n_i/2$. By specifying the fusion, we avoid duplication in the results. For example, one would obtain an equivalent list of (sister) candidates if fusing adjacency in Γ_1, non-adjacency in Γ_2, and incidence in the srd.

3.1. **Parameters.** In Section 4 of [12], Higman shows that all srd parameters (and the srg parameters of Γ_i) are determined by $\{n_1, n_2, S_1, a_1, b_1, b_2\}$. Here we supply formulas for the srd parameters as determined by $\{n_i, k_i, \lambda_i, \mu_i\}$ **assuming** that we have an srd with fusion to srg Γ_0 as in the lemma.

$$\begin{aligned} S_1 &= k_0 - k_2 & S_2 &= k_0 - k_1 \\ a_1 &= \lambda_0 - \lambda_2 & a_2 &= \lambda_0 - \lambda_1 \\ b_1 &= \mu_0 - \mu_2 & b_2 &= \mu_0 - \mu_1 \\ N_1 &= \frac{a_2 k_1}{S_2} & N_2 &= \frac{a_1 k_2}{S_1} \\ P_1 &= \frac{(k_1 - N_1) S_1}{n_1 - S_1} & P_2 &= \frac{(k_2 - N_2) S_2}{n_2 - S_2} \\ \rho_1 &= N_1 - P_1 & \rho_2 &= N_2 - P_2 \\ \sigma_1 &= -\frac{S_2 - b_2}{a_2 - b_2} & \sigma_2 &= -\frac{S_1 - b_1}{a_1 - b_1} \end{aligned} \tag{1}$$

It is not assumed that $\rho_i > 0$, as there are examples of both $N_1 > P_1$ and $N_1 < P_1$. That is, we permit σ_i to be the positive eigenvalue of Γ_i.

4. Equivalence of certain srd's with strongly regular decompositions

Theorem 2.8 of [9] gives conditions on the parameters of Γ_1 and Γ_2 under which a primitive srg Γ_0 admitting a proper, non exceptional, strongly regular decomposition (Γ_1, Γ_2) affords a strongly regular design with Γ_1 as point graph and Γ_2 as block graph. In what follows, we derive equivalent conditions from the assumption of a nontrivial srd with fusion to an srg with strongly regular decomposition as in Section 3.

Lemma 4.1. *An srd*$(n_i, S_i, a_i, b_i, N_i, P_i)$ *with fusion to srg*$(n_1+n_2, k_0, \lambda_0, \mu_0)$ *satisfies*

(i) $a_1 - b_1 = \rho_1 - \sigma_2$

(ii) $a_2 - b_2 = \rho_2 - \sigma_1$.

Proof.

$$\begin{aligned}
\sigma_2 + a_1 - b_1 &= \sigma_2 + (\lambda_0 - \mu_0) - (\lambda_2 - \mu_2) && \text{by (ii) and (iii) of Lemma 3.1} \\
&= \sigma_2 + (\lambda_0 - \mu_0) - (\rho_2 + \sigma_2) && \text{by standard srg relations} \\
&= (N_1 - P_1) + (N_2 - P_2) - \rho_2 && \text{by (ii) and (iii) of Lemma 3.1} \\
&= \rho_1 && \text{by (1).}
\end{aligned} \tag{2}$$

A similar calculation shows that $\rho_2 = \sigma_1 + a_2 - b_2$. □

Lemma 4.2. *The eigenvalues of Γ_0, a strongly regular graph obtained from a strongly regular design as in Lemma 3.1, are $k_0, \sigma_1, \rho_1 + \rho_2 - \sigma_1$.*

Proof. As k_0 is the valency of Γ_0, it is an eigenvalue of multiplicity 1. The other eigenvalues r and s satisfy $\lambda_0 - \mu_0 = r + s$ and $k_0 - \mu_0 = -rs$. Since $\rho_i = N_i - P_i$, we obtain

$$\rho_1 + \rho_2 = \lambda_1 + a_2 - (\mu_1 + b_2) = \lambda_0 - \mu_0 = r + s$$

using Lemma 3.1. Now

$$\begin{aligned}
\sigma_1 &= -\frac{S_2 - b_2}{a_2 - b_2} \\
&= -\frac{(k_0 - k_1) - (\mu_0 - \mu_1)}{\rho_1 + \rho_2 - (\lambda_1 - \mu_1)} \\
&= \frac{-(k_0 - \mu_0) - (k_1 - \mu_1)}{\rho_1 + \rho_2 - (\rho_1 + \sigma_1)} \\
&= \frac{rs - \rho_1 \sigma_1}{\rho_2 - \sigma_1}
\end{aligned} \tag{3}$$

thus $\sigma_1(\rho_1 + \rho_2 - \sigma_1) = rs$. But then $\sigma_1(r + s - \sigma_1) = rs$ implying $(r - \sigma_1)(s - \sigma_1) = 0$. We conclude $\{r, s\} = \{\sigma_1, \rho_1 + \rho_2 - \sigma_1\}$. □

Corollary 4.1. *σ_2 is an eigenvalue of Γ_0 different from k_0.*

Proof. We aim to show $\sigma_2 \in \{r, s\}$. The proof above shows $\sigma_1(\rho_1+\rho_2-\sigma_1) = rs$. An identical calculation, with only the the subscripts changed, gives $\sigma_2(\rho_1 + \rho_2 - \sigma_2) = rs$. We conclude that σ_2 and $\rho_1+\rho_2-\sigma_2$ are two (rational) numbers with sum equal to $r+s$ and product equal to rs. This uniquely determines $\{\sigma_2, \rho_1 + \rho_2 - \sigma_2\} = \{r, s\} = \{\sigma_1, \rho_1 + \rho_2 - \sigma_1\}$. □

Note that it also follows that either $\sigma_2 = \sigma_1$ or $\sigma_1 + \sigma_2 = \rho_1 + \rho_2$.

Lemma 4.3. *$k_1 - S_1$ is an eigenvalue of Γ_0 different from k_0.*

Proof. We must show $k_1 - S_1 \in \{\sigma_1, \rho_1 + \rho_2 - \sigma_1\}$. The adjacency matrix of Γ_0 has the form

$$\begin{bmatrix} A_1 & C \\ C^T & A_2 \end{bmatrix}$$

where A_i is the adjacency matrix of Γ_i and C is the incidence matrix of the strongly regular design. As C has row sum S_1 and column sum S_2, we obtain an eigenvector with eigenvalue $k_1 - S_1$ taking j_i to be the all-ones vector of length n_i:

$$\begin{bmatrix} A_1 & C \\ C^T & A_2 \end{bmatrix} \begin{bmatrix} j_1 \\ -j_2 \end{bmatrix} = \begin{bmatrix} A_1 j_1 - C j_2 \\ C^T j_1 - A_2 j_2 \end{bmatrix} = \begin{bmatrix} (k_1 - S_1) j_1 \\ -(k_2 - S_2) j_2 \end{bmatrix}.$$

The result follows from (i) of 3.1. □

Because $\sigma_1 \in \{r, s\}$ we see that the strongly regular decomposition of Γ_0 into Γ_1 and Γ_2 is not exceptional. That is, as in [9], the assumption that a strongly regular decomposition arises from a strongly regular design implies that the srd is not of exceptional type.

5. Symmetry and rank 5 fusions

Proposition 2 of [16] states that in the case of a symmetric srd, the fusion according to $\pi = \{1,4\}\{2,5\}\{3,6\}\{7,9\}\{8,10\}$ yields a symmetric, rank 5 association scheme. This scheme is necessarily imprimitive as the second and third relations are clearly disconnected. Indeed, by direct computation as in Section 2, we find the fusion feasible if and only if $n_1 = n_2$, $k_1 = k_2$, $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$, and likewise for the remaining srd parameters a_i, b_i, S_i, N_i, P_i. This proves the following lemma.

Lemma 5.1. *A strongly regular design fuses to a rank 5 symmetric cc if and only if the srd is symmetric. Equivalently, the rank 5 fusion according to the partition $\pi = \{1,4\}\{2,5\}\{3,6\}\{7,9\}\{8,10\}$ is feasible if and only if Γ_1 and Γ_2 have the same parameters.*

Proof. The second part of the statement follows from 5.1 in [12] which states: If $n_1 = n_2$ (in an srd), then any one of (i) $a_1 = a_2$, (ii) $b_1 = b_2$, (iii) $k_1 = k_2$ and $r_1 - r_2$ implies that the srd is symmetric. □

These schemes are precisely the cometric, Q-antipodal, 4-class schemes of [22], Section 7.5.

6. Feasible parameter sets

Table 1 shows strongly regular decompositions to $n_1, n_2 \leq 400$, extending Table 1 of [9], and including the srd parameters. The parameter sets we refer to in the comments are from [12] in the case of "DGH", [9] in the case of "HH", and [15] in the case of "KR".

6.1. **Nonexistence.** Parameter sets 12 and 17 are strongly regular decompositions that do not exist, due to [9] and therefore these SRDs do not exist. Set 17 is, in addition, ruled out by the Krein condition of Section 2.

6.2. **Constructions.**

6.2.1. *Quasi-symmetric 3-designs.* Sets #5 and #6 in the table of parameters arise from quasi-symmetric 3-designs (See [20] and the references therein). These are $3-(v,k,\lambda)$ designs with two block intersection sizes. Because a 3-design is also a 2-design, it is well known that the block graph of such a design is strongly regular. Letting Γ_0 be the block graph of a quasi-symmetric 3-design, we obtain a strongly regular decomposition by fixing a point x and partitioning the set of blocks into those containing x and those not containing x. The block graphs of the derived and residual designs that result from removing x are also strongly regular. They are possibly complete or null, however, depending on whether the designs are properly quasi-symmetric, or are in fact symmetric (having one block intersection size). In case either of these is symmetric, the strongly regular decomposition is improper, the srd is imprimitive, and therefore does not appear in our table. This is the case for the Witt design $4-(23,7,1)$, which is #8 in the table of [9]. The derived and residual designs are $3-(22,6,1)$, accounting for set #5, and $3-(22,7,4)$, set #6. It is conjectured that quasi-symmetric 3-designs are few and far between, with the complete list including Hadamard 3-designs (which lead to imprimitive srd's) in addition to those named above.

6.2.2. *Symplectic graphs.* The construction of Example 3.5 in [9] involves the symplectic graphs. Two infinite families of srds are afforded by these strongly regular decompositions, having the parameters given below in the form (n,k,r,s).

$$\Gamma_0 : \left(2^{2m-1}, 2^{2m-1}-2, 2^{m-1}-1, -2^{m-1}-1\right)$$
$$\Gamma_1 : \left(2^{2m-1}+2^{m-1}-1, 2^{2m-2}+2^{m-1}-2, 2^{m-1}-1, -2^{m-2}-1\right)$$
$$\Gamma_2 : \left(2^{2m-1}-2^{m-1}, 2^{2m-2}-1, 2^{m-2}-1, -2^{m-1}-1\right)$$

Γ_0 : as above

$$\Gamma_1 : \left(2^{2m-1}-2^{m-1}-1, 2^{2m-2}-2^{m-1}-2, 2^{m-2}-1, -2^{m-1}-1\right)$$
$$\Gamma_2 : \left(2^{2m-1}+2^{m-1}, 2^{2m-2}-1, 2^{m-1}-1, -2^{m-2}-1\right)$$

Sets #1, #2, #9, and #11 are accounted for by these two families. The next two symplectic examples, shown below, are beyond the reach of our table of parameters.

n / n_i	k / k_i	λ / λ_i	μ / μ_i	r / ρ_i	s / σ_i	S_i	a_i	b_i	N_i	P_i
1023	510	253	255	15.0	-17.0					
527	270	141	135	-9	15	255	127	119	126	135
496	255	126	136	7	-17	240	112	120	127	120
1023	510	253	255	15.0	-17.0					
495	238	109	119	7	-17	255	127	135	126	119
528	255	126	120	-9	15	272	144	136	127	136

6.2.3. *Hemisystems.* A family of strongly regular decompositions with parameters shown below arises from a hemisystem of a generalized quadrangle of order (q^2, q). Briefly, a generalized quadrangle is a finite point-line incidence geometry $\mathsf{GQ}(s,t)$ in which each line is incident with $s+1$ points and each point with $t+1$ lines; any two points are incident with at most one line; and for every nonincident point P and line $\mathcal{L}$ there is exactly one line on P that meets $\mathcal{L}$. The classical Hermitian polar space $\mathcal{H}(3,q^2)$ induced by a non-degenerate unitary form on the projective geometry $\mathsf{PG}(3,q^2)$ forms a $\mathsf{GQ}(q^2,q)$. A hemisystem in such a geometry is a fixed set of lines that contains exactly half of the lines on any one point. Existence of a hemisystem in $\mathsf{GQ}(q^2,q)$ thus necessitates q odd. Furthermore, the complementary set of lines is clearly also a hemisystem.

It is well known that the line graph of $\mathsf{GQ}(t^2,t)$ is strongly regular where two lines are adjacent if and only if they have a point in common. It is shown in [4] that the point graph associated with a dual hemisystem (a set of points rather than lines) is strongly regular. We therefore have, in a $\mathsf{GQ}(q,q^2)$, that a hemisystem determines a strongly regular decomposition of the line graph, with Γ_1 and Γ_2 having the same parameters. The parameters are given as (n,k,λ,μ) and are taken from [6].

$$\begin{aligned} \Gamma_0 : &\quad \mathsf{srg}\left((t^3+1)(t+1), t(t^2+1), t-1, t^2+1\right) \\ \Gamma_1 = \Gamma_2 : &\quad \mathsf{srg}\left((t^3+1)(t+1)/2,\ (t^2+1)(t-1)/2,\ (t-3)/2,\ (t-1)^2/2\right) \end{aligned}$$

Although it was thought for about 40 years that very few hemisystems existed, there has been much work in recent years on these and other structures derived from finite classical polar spaces and a number of infinite families are now known. The survey paper [6] details this recent work and along with the references therein provides ample background material. In particular, Bamberg, Giudici and Royle showed that every *flock generalised quadrangle* of order (q^2,q) has a hemisystem ([1],[2]) and observed that hemisystems “actually exist in great profusion”. For our purposes we only scratch the surface of this topic to note that parameter sets #4 and #21 ($t=3$ and $t=5$) are in this family and that the next instance ($t=7$) has $n=2752$.

The associated srd's have sisters as shown below, using the complement of both Γ_1 and Γ_2 as is necessary for the Krein condition. Of note, Hobart's bound is attained for these examples. That is, $XY - ZW = 0$ in the notation of Section 2.2.

$$S_i = \frac{(t^2+1)(t+1)}{2} \qquad N_i = \frac{t^2(t+1)}{2}$$
$$a_i = \frac{(t+1)^2}{2} \qquad P_i = \frac{t(t^2+1)}{2}$$
$$b_i = \frac{t+1}{2}$$

Because these srd's are symmetric, Section 5 applies and we find examples of cometric, Q-antipodal association schemes of rank 5 related to these hemisystems, as has been noted elsewhere.

6.3. **Comments.**

(1) Set #2 is a sister to #39 but not #38 on the Klin and Reichard list – that one does not satisfy the conditions for a strongly regular decomposition.
(2) Set #3 is the Higman-Sims group example.
(3) Set #7 is a group case involving the McLaughlin graph (see [8]).
(4) Set #8 is mentioned in [9] but existence is unknown.

7. Strongly regular decompositions of the complete graph

Strongly regular decompositions of the complete graph have been investigated by Kharaghani et. al., ([14]), Momihara and Okumura ([18]), and van Dam ([21]), among others. In the setting of Section 2, we now consider the case in which Γ_0 is complete with srg parameters $(n = n_1 + n_2, n - 1, n - 2, 0)$.

Lemma 7.1. *An srd has fusion to a strongly regular decomposition in which Γ_0 is a complete graph if and only if both constituent graphs are complete and the incidence structure is complete bipartite.*

Proof. Since $\mu_0 = 0$, we get $\mu_i = 0$ and $b_i = 0$ which forces $P_i = 0$. It follows from the conditions in Section 2.1 that $N_i = k_i$, since $S_i \neq 0$. But then $\rho_i = k_i$ and $a_i = S_i$, whence $\sigma_i = -1$. This is the situation described in [12] as an srd obtained by repeating points of a quasi-symmetric design. It follows further, however, from the proof of Lemma 4 that $\rho_1 + \rho_2 = \lambda_0 - \mu_0$. By assumption, $\lambda_0 = n_1 + n_2 - 2$ and $\mu_0 = 0$, giving $\rho_1 + \rho_2 = n_1 + n_2 - 2$. But $\rho_i = k_i$, hence $k_1 + k_2 = n_1 + n_2 - 2$. Now, $k_i \leq n_i - 1$ always, so $k_i = n_i - 1$ is the only possibility. We now see that both Γ_1 and Γ_2 are complete graphs, and that the srd must satisfy

$$b_i = 0, \qquad P_i = 0, \qquad N_i = k_i = n_i - 1, \qquad \text{and} \qquad a_i = S_i. \tag{4}$$

Finally, by (6) of 3.2 in [12], $(a_1 - n_1)a_2 = (a_2 - n_2)a_1 = 0$ which implies $a_i = n_i$. This shows that each block of the srd is incident with all n_1 points, and each point lies in all n_2 blocks. The incidence structure is that of a complete bipartite graph. □

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Table 1: Strongly regular designs with strongly regular decomposition

	n	k	λ	μ	r	s							
	n_i	k_i	λ_i	μ_i	ρ_i	σ_i	S_i	a_i	b_i	N_i	P_i	$\exists\Gamma_i$	Comments
	63	30	13	15	3.0	-5.0						+	
1	27	10	1	5	1	-5	15	7	9	6	5	!	Sister to DGH #2,
	36	15	6	6	-3	3	20	12	10	7	10	+	HH #4
	63	32	16	16	4.0	-4.0						+	Sister to DGH #4,
2	28	12	6	4	-2	4	16	10	8	6	8	+	HH #3,
	35	16	6	8	2	-4	20	10	12	10	8	+	sister to KR #39
	100	22	0	6	2.0	-8.0						+	
3	50	7	0	1	-3	2	15	0	5	0	3	!	Sister to DGH #13,
	50	7	0	1	-3	2	15	0	5	0	3	!	HH #12
	112	30	2	10	2.0	-10.0						!	
4	56	10	0	2	-4	2	20	2	8	1	5	!	HH #13
	56	10	0	2	-4	2	20	2	8	1	5	!	
	176	70	18	34	2.0	-18.0						!	
5	56	10	0	2	-4	2	28	10	16	3	7	!	HH #16
	120	42	8	18	-12	2	60	18	32	15	27	!	
	253	112	36	60	2.0	-26.0						+	
6	77	16	0	4	-6	2	42	18	26	6	12	!	HH #24
	176	70	18	34	-18	2	96	36	56	30	48	!	
	162	56	10	24	2.0	-16.0						!	
7	81	20	1	6	-7	2	36	9	18	5	12	!	HH #15
	81	20	1	6	-7	2	36	9	18	5	12	!	
	265	96	32	36	6.0	-10.0						?	
8	105	32	4	12	2	-10	42	14	18	14	12	!	HH #28
	160	54	18	18	-6	6	64	28	24	18	24	?	
	255	126	61	63	7.0	-9.0						+	
9	119	54	21	27	3	-9	63	31	35	30	27	+	HH #26
	136	63	30	28	-5	7	72	40	36	31	36	+	
	340	108	30	36	6.0	-12.0						?	
10	120	42	8	18	2	-12	36	8	12	14	12	!	
	220	72	22	24	-8	6	66	22	18	16	24	?	
	255	128	64	64	8.0	-8.0						+	
11	120	56	28	24	-4	8	64	36	32	28	32	+	HH #25
	135	64	28	32	4	-8	72	36	40	36	32	+	
	324	57	0	12	3.0	-15.0						-	
12	162	21	0	3	-6	3	36	0	9	0	6	?	Γ_0 DNE
	162	21	0	3	-6	3	36	0	9	0	6	?	
	406	165	68	66	11.0	-9.0						?	
13	175	66	29	22	-4	11	75	35	30	26	30	?	
	231	90	33	36	6	-9	99	39	44	42	36	?	
	399	198	97	99	9.0	-11.0						+	
14	189	88	37	44	4	-11	99	49	54	48	44	?	
	210	99	48	45	-6	9	110	60	55	49	55	+	
	399	200	100	100	10.0	-10.0						+	
15	190	90	45	40	-5	10	100	55	50	45	50	?	
	209	100	45	50	5	-10	110	55	60	55	50	+	
	392	115	18	40	3.0	-25.0						?	

16	196	45	4	12	-11	3	70	14	28	9	20	?	
	196	45	4	12	-11	3	70	14	28	9	20	?	
	486	165	36	66	3.0	-33.0						-	
17	243	66	9	21	-15	3	99	27	45	18	33	?	Γ_0 DNE
	243	66	9	21	-15	3	99	27	45	18	33	?	
	576	120	28	24	12.0	-8.0						?	pg(15,7,3)
18	225	42	15	6	-3	12	50	15	10	7	10	$+$	OA(15,3)
	351	70	13	14	7	-8	78	13	18	21	14	?	
	640	243	66	108	3.0	-45.0						?	
19	320	99	18	36	-21	3	144	48	72	33	54	?	
	320	99	18	36	-21	3	144	48	72	33	54	?	
	750	210	55	60	10.0	-15.0						?	pg(14,14,4)
20	375	110	25	35	5	-15	100	30	25	33	28	?	
	375	110	25	35	5	-15	100	30	25	33	28	?	
	756	130	4	26	4.0	-26.0						$+$	GQ$(5,5^2)$; $O^-(6,5)$ polar graph;
21	378	52	1	8	-11	4	78	3	18	2	13	$+$	hemisystem in PG$(3,5^2)$
	378	52	1	8	-11	4	78	3	18	2	13	$+$	
	784	116	0	20	4.0	-24.0						?	
22	392	46	0	6	-10	4	70	0	14	0	10	?	
	392	46	0	6	-10	4	70	0	14	0	10	?	
	800	204	28	60	4.0	-36.0						?	
23	400	84	8	20	-16	4	120	20	40	14	30	?	
	400	84	8	20	-16	4	120	20	40	14	30	?	

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