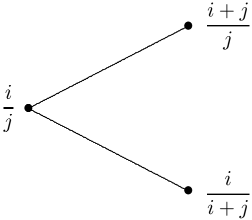
The Tree of All Fractions

UNB Math Camp, 2015

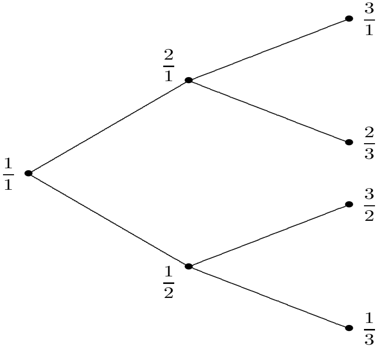
Outline

The tree of all fractions

We begin with the fraction $\frac{1}{1}$. Each fraction has two children:

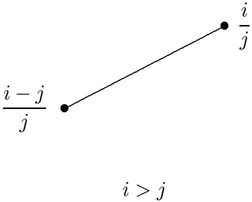


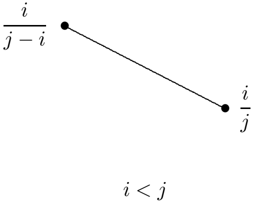
The first three levels



Parents

To go backwards:





Observations

- Every fraction occurs in the tree.
- No fraction occurs more than once in the tree. (What about $\frac{4}{6}$ and other equivalents?)
- So, every fraction occupies a *unique* position in the tree.

One-to-one correspondence

Every path in the tree ends at some fraction, and for every (reduced, positive) fraction, there is a path in the tree.

This means we have a *one-to-one correspondence* between the *paths* in the tree and the *positive rational numbers*.

Binary numerals

- Represent the paths in the tree by *binary numerals*.
- Start with a 1 indicating the top of the tree $\left(\frac{1}{1}\right)$.
- Append a 1 for "up" and a 0 for "down".

Example

Find the path to $\frac{5}{9}$ and represent it as a binary numeral.

Example

Find the fraction that corresponds to the whole number 42.

Exercises

Definition (Notation:)

Let F be the function from the whole numbers to the rational numbers, determined by the labelling of paths in the tree. For example, $F(12) = 5/9$ and $F(42) = 8/13$.

1. Find $F(25)$.
2. Solve $F(n) = 4/7$ (find n).

The big idea

We have defined a one-to-one correspondence between two sets:

$$\text{positive rational numbers} \longleftrightarrow \text{positive integers}$$

This shows that those sets are *conumerous*.

The real numbers are *uncountably infinite* while the rationals are *countably infinite*.

More exercises

Use the notation $F(n) =$ the fraction that corresponds to n. For example, $F(3) = \frac{2}{1}$.

1. Find (a) $F(19)$ (b) $F(100)$ (c) $F(63)$.
2. Find n such that $F(n)$ is (a) $\frac{7}{11}$ (b) $\frac{13}{18}$ (c) $\frac{29}{42}$.
3. What is the characteristic property of binary numerals that represent even numbers? Use this to describe the fractions that correspond (through F) to even numbers.
4. Each fraction has two children. Using the correspondence, how would you describe the "children" of each positive integer? For instance, 5 is 101 in binary form. This corresponds to the fraction 3/2. It's children are 5/2 and 3/5, which correspond to 11 (1011 in binary) and 10 (1010 in binary) respectively. Thus the "children" of 5 are 11 and 10. See if you can come up with a quick rule for finding children. What are the children of 57? 102? 2015?
5. Describe which fractions correspond to multiples of 4.

References

N. Calkin and H. Wilf
Recounting the rationals
American Mathematical Monthly, 107 (2000), 360–363.

W. Lindgren, A. Sankey, and G. Roberts
Introduction to Mathematical Thinking, course notes, 1999.