How GPS Works

March 2, 2017

Outline

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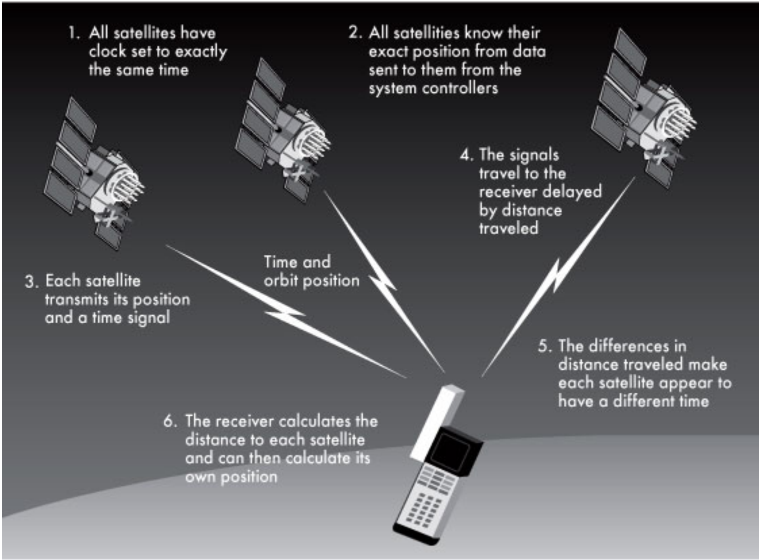


Image from aero.org: www.aero.org/education/primers/gps/images/5steps-illustration.jpg

The problem

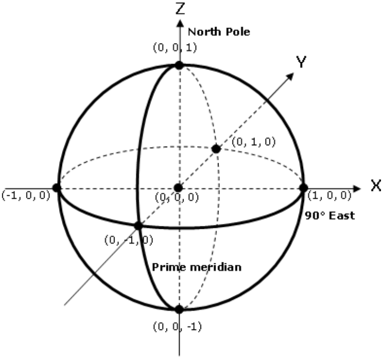
How does the GPS determine the location of the receiver at a particular point in time?

- We have the position and time from each of four satellites.
- We need to find the position of the GPS receiver.
- We know that the signal travels at the speed of light: about 0.47 Earth radii per hundredth of a second.

The unit circle

Locations are given using the x, y, z coordinate system.

- The Earth is a unit sphere, centred at the origin.
- Points on the surface of the earth satisfy the equation $x^2 + y^2 + z^2 = 1$.



http://resources.esri.com/help/9.3/ArcGISEngine/dotnet/bitmaps/b0e91ce8-c180-47dc-8323-06cac5d770641.png

Satellite data

Satellite	Position	Time
1	$(1.11, 2.55, 2.14)$	1.29
2	$(2.87, 0.00, 1.43)$	1.31
3	$(0.00, 1.08, 2.29)$	2.75
4	$(1.54, 1.01, 1.23)$	4.06

Distance

The signal from Satellite 1 was sent at time 1.29, and received at time t. (We do not assume that we know t with enough accuracy.) Thus the time it travelled was $t - 1.29$ hundredths of a second.

The distance it travelled is:

$$d = 0.47(t - 1.29) \text{ Earth radii.}$$

Equating distances

The distance can also be computed using the distance formula: the distance between the point $(1.11, 2.55, 2.14)$ and the point (x, y, z) is

$$d = \sqrt{(x - 1.11)^2 + (y - 2.55)^2 + (z - 2.14)^2}.$$

Since this must equal the distance we calculated using the time, we have

$$\sqrt{(x - 1.11)^2 + (y - 2.55)^2 + (z - 2.14)^2} = 0.47(t - 1.29).$$

Simplify

Square both sides and simplify to get:

$$2.22x + 5.10y + 4.28z - 0.57t = x^2 + y^2 + z^2 - 0.22t^2 + 11.95.$$

Similarly, we can get equations for satellites 2, 3, 4:

$$\begin{aligned} 5.74x + \qquad\qquad + 2.86z - 0.58t &= x^2 + y^2 + z^2 - 0.22t^2 + 9.90 \\ 2.16y + 4.58z - 1.21t &= x^2 + y^2 + z^2 - 0.22t^2 + 4.74 \\ 3.08x + 2.02y + 2.46z - 1.79t &= x^2 + y^2 + z^2 - 0.22t^2 + 1.26 \end{aligned}$$

Linear?

At this point, we have a system of equations, but they are not *linear* equations. However, we can simplify to a linear system by subtracting.

System of equations

$$\begin{array}{rcrcrcrcr} 3.52x & - & 5.10y & - & 1.42z & - & 0.01t & = & 2.05 \\ -2.22x & - & 2.94y & + & 0.30z & - & 0.64t & = & 7.21 \\ 0.86x & - & 3.08y & - & 1.82z & - & 1.22t & = & -10.69 \end{array}$$

This can be solved by substitution, elimination, or matrix methods.

$$\left[\begin{array}{rrrr|r} 3.52 & -5.10 & -1.42 & -0.01 & 2.05 \\ -2.22 & -2.94 & 0.30 & -0.64 & 7.21 \\ 0.86 & -3.08 & -1.82 & -1.22 & -10.69 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.36 & 2.97 \\ 0 & 1 & 0 & 0.03 & 0.81 \\ 0 & 0 & 1 & 0.79 & 5.91 \end{array}\right]$$

Solution

$$\begin{aligned} x &= 2.97 - 0.36t \\ y &= 0.81 - 0.03t \\ z &= 5.91 - 0.70t \end{aligned}$$

Putting this back into the original first equation gives

$$0.54t^2 - 6.65t + 20.32 = 0$$

with solutions $t = 6.74$ and $t = 5.60$, so

$$(x, y, z) = (0.55, 0.61, 0.56) \text{ or } (x, y, z) = (0.96, 0.65, 1.46.$$

The second solution is not on the unit sphere; the first one is. This is then converted into latitude and longitude, but we'll save that for another day!

References I

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